UNIVERSITY OF MONTPELLIER

Non-Linear Vlasov Equation with Logarithmic Non-Linearity

A THESIS SUBMITTED IN PARTIAL FULFILLMENT FOR THE DEGREE OF MASTER

MASTER AMS, UNIVERSITY OF PARIS-SUD

Author:
Guillaume FERRIERE

Supervisors:
Rémi CARLES
Matthieu HILLAIRET

September 6, 2018

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Abstract

In this survey, we briefly review some results about the universal dynamics for the defocusing non-linear Schrödinger equation with logarithmic non-linearity ([5]) without semi-classical constant and we extend it to the case with semi-classical constant. Such results allow us to pass to the limit when the semi-classical constant tends to 0 thanks to the Wigner Transform and the Wigner Measure, and give us the idea to get similar results for the non-linear Vlasov equation with logarithmic non-linearity.

Acknowledgements

I would like to express my deep gratitude to Rémi Carles and Matthieu Hillairet, my research supervisors, for their patient guidance and useful critiques of this research work. Their precious advice were always enlightening and motivating. I would also like to thank Kléber Carrapatoso for his advice and assistance, in particular for the properties of some limits. My grateful thanks are also extended to my "office room-mates" during this period: André Harnist, Gwenaël Peltier and Antoine Toussaint for their help and the mutual support we provided to each other.

I would also like to extend my thanks to the researchers and technicians of IMAG for having warmly welcomed me in the laboratory and for their help in offering me the resources needed for this work and for the PhD which follows. I am also grateful towards the teachers of the Master AMS, whose courses were very interesting and helpful in the view of this work.

Finally, I wish to thank my parents, my sister, my relatives and my friends for their support and encouragement throughout my studies.

Chapter 1

Introduction and Main Results

1.1 Introduction

This survey is concerned by the large time behaviour of solutions $f = f(t, x, \xi)$ for the Vlasov equation

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f - \lambda \nabla_x (\ln \rho) \cdot \nabla_\xi f = 0 & t > 0, (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d, \\ f(0, x, \xi) = f_{in}(x, \xi) & (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $\lambda > 0$ and $\rho(t, x) = \int_{\mathbb{R}^d} f(t, x, d\xi)$. Such an equation arises in plasma physics, e.g. for quasi-neutral plasmas in the core or tokamaks when one focuses on the direction of the magnetic lines by assuming the electrons to be adiabatic ([9, 7]). There, the equation appears in dimension $d = 1$ and f denotes the ionic distribution function.

Due to the derivative of the density ρ with respect to space in the force term $\nabla_x(\ln \rho)$, this equation is highly singular. The Cauchy problem is still very difficult to face and, for the moment, it has only been proven to be well-posed for very specific initial data. However, this equation has a link with the Isothermal Euler System when we consider mono-kinetic functions of (1.1) of the form

$$f_{in}(x, \xi) = \rho_{in}(x) dx \otimes \delta_{\xi = v_{in}(x)},$$

with time-dependent mono-kinetic solutions of the form

$$f(t, x, \xi) = \rho(t, x) dx \otimes \delta_{\xi = v(t,x)},$$

because ρ and v have then to solve the isothermal Euler system

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho v) = 0, \\ \partial_t (\rho v) + \nabla_x \cdot (\rho v \otimes v) + \lambda \nabla_x \rho = 0. \end{cases} \tag{1.2}$$

Such a system relates to another equation: a Schrödinger equation. Like before, the potential in this equation is logarithmic. Formally, for $\varepsilon > 0$, consider

$$i \varepsilon \partial_t u_\varepsilon + \frac{\varepsilon^2}{2} \Delta u_\varepsilon = \lambda u_\varepsilon \ln |u_\varepsilon|^2, \qquad u_\varepsilon(0, .) = u_{\varepsilon, in}. \tag{1.3}$$

Any function $u_\varepsilon = a_\varepsilon e^{i \frac{\phi_\varepsilon}{\varepsilon}}$ where $(t, x) \mapsto a_\varepsilon(t, x) \in \mathbb{C}^d$ and $(t, x) \mapsto \phi_\varepsilon(t, x) \in \mathbb{R}$ are solutions to

$$\begin{cases} \partial_t v_\varepsilon + (v_\varepsilon \cdot \nabla) v_\varepsilon + \lambda \nabla (\ln |a_\varepsilon|^2) = 0, & v_\varepsilon(0, x) = \phi'_{in}(x), \\ \partial_t a_\varepsilon + v_\varepsilon \cdot \nabla a_\varepsilon + \frac{a_\varepsilon}{2} \nabla \cdot v_\varepsilon = i \frac{\varepsilon}{2} \Delta a_\varepsilon, & a_\varepsilon(0, x) = a_{in}(x), \end{cases} \tag{1.4}$$

with the relations

$$\phi_\varepsilon(t,x) = \phi_{in}(x) - \int_0^t \left(\frac{1}{2}|v_\varepsilon(\tau,x)|^2 + \lambda \ln |a_\varepsilon(\tau,x)|^2\right) d\tau, \qquad v_\varepsilon(t,x) = \nabla_x \phi_\varepsilon(t,x), \tag{1.5}$$

is a solution to (1.3) with $u_{\varepsilon,in} = \sqrt{\rho_{in}}\, e^{i\,\frac{\phi_{in}}{\varepsilon}}$. When we formally pass to the limit $\varepsilon \to 0$ in (1.4), it yields

$$\begin{cases} \partial_t v + (v \cdot \nabla)\, v + \lambda \nabla(\ln |a|^2) = 0, \qquad v(0,x) = \nabla \phi_{in}(x), \\ \partial_t a + v \cdot \nabla a + \dfrac{a}{2} \nabla \cdot v = 0, \qquad a(0,x) = a_{in}(x), \end{cases} \tag{1.6}$$

which is the symmetrized version of (1.2) with $\rho = |a|^2$.

The Wigner Transform is one of the available tools we have to rigorously relate a Schrödinger equation to the linked Vlasov equation (with same potential). It revealed good properties to make rigorous the previous formal link, known as semi-classical limit in physics. Defined by

$$W_\varepsilon(x,\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi\cdot z} u_\varepsilon\left(x + \frac{\varepsilon z}{2}\right) \overline{u_\varepsilon\left(x - \frac{\varepsilon z}{2}\right)}\, dz, \tag{1.7}$$

this tool transforms a function $u_\varepsilon \in L^2(\mathbb{R}^d)$ (which can also be time-dependent, in that case the Wigner Transform is also time-dependent) into a function W_ε called the Wigner Transform of u_ε defined on the phase space. This function usually converges (in a suitable sense) to a measure (called Wigner Measure) solution to the Vlasov equation

$$\partial_t f + \xi \cdot \nabla_x f - \nabla_x V_0 \cdot \nabla_\xi f = 0,$$

when it is linked to u_ε solution of the corresponding Schrödinger equation

$$i\varepsilon\, \partial_t u_\varepsilon + \frac{\varepsilon^2}{2} \Delta u_\varepsilon = V_0 u_\varepsilon, \tag{1.8}$$

when ε tends to 0 as soon as V_0 verifies some suitable properties. Moreover, this approach allows to consider a different framework from mono-kinetic solutions.

1.2 Motivations

The main interest of this survey starts with the article of R. Carles and I. Gallagher [5]. Not only it provides well-posedness of the Cauchy problem for the Logarithmic Schrödinger Equation for initial data in some Sobolev space, but they also produce interesting results regarding the behaviour of such a solution.

Theorem 1.1 ([5, Theorems 1.5. and 1.7.]). *Let $\lambda > 0$, $u_{in} \in \mathcal{F}(H^1) \cap H^1(\mathbb{R}^d)$. Then there exists a unique, global solution $u \in L^\infty_{loc}(\mathbb{R}, \mathcal{F}(H^1) \cap H^1(\mathbb{R}^d))$ of*

$$i\, \partial_t u + \frac{\Delta u}{2} = \lambda u \ln |u|^2, \qquad u(0,.) = u_{in}. \tag{1.9}$$

Moreover, $u \in \mathcal{C}(\mathbb{R}, L^2 \cap H^1_w(\mathbb{R}^d))$. Assume $u_{in} \neq 0$ and rescale this solution to $v = v(t,y)$ by setting

$$u(t,x) = \frac{1}{\tau(t)^{\frac{d}{2}}} \frac{||u_{in}||_{L^2}}{||\gamma||_{L^2}}\, v\left(t, \frac{x}{\tau(t)}\right) e^{i\,\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}},$$

where $\gamma(x) = e^{-\frac{|x|^2}{2}}$ *and* τ *is the solution of*

$$\ddot{\tau} = \frac{2\lambda}{\tau}, \qquad \tau(0) = 1, \qquad \dot{\tau}(0) = 0. \tag{1.10}$$

Then $\tau \in \mathcal{C}^\infty(\mathbb{R}^+)$ *and there exists* C *such that for all* $t \geq 0$,

$$\int_{\mathbb{R}^d} \left(1 + |y|^2 + \left|\ln |v(t,y)|^2\right|\right) |v(t,y)|^2 \, dy + \frac{1}{\tau(t)^2} ||\nabla_y v(t)||_{L^2(\mathbb{R}^d)} \leq C.$$

We have moreover

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} |v(t,y)|^2 \, dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \gamma^2(y) \, dy.$$

Finally,

$$|v(t,.)|^2 \underset{t\to\infty}{\rightharpoonup} \gamma^2 \qquad \text{weakly in } L^1(\mathbb{R}^d).$$

Two new features characterizing the dynamics associated to (1.9) are interesting:

- Thanks to the estimate [5, Lemma 1.6.], the dispersion rate is in $\left(t\sqrt{\ln t}\right)^{\frac{d}{2}}$. Usually in $t^{\frac{d}{2}}$ for the Schrödinger equation, it is altered by a logarithmic factor due to the non-linearity of the equation, accelerating the dispersion.
- Up to a rescaling, the modulus of the solution weakly converges for large time to a universal Gaussian profile.

The existence and uniqueness of the solution can then be easily proved for the general case $\varepsilon > 0$, but we wonder if the other properties can be generalized with some uniformity in ε and, if so, if the semi-classical limit propagates such a behaviour. As previously said, the Wigner Transform may be the tool to make the link we are looking for. This intuition is strengthened by the view of the following result from R. Carles and A. Nouri [7], recalling the definition of Zhidkov spaces:

$$X^s(\mathbb{R}) = \left\{ f \in L^\infty(\mathbb{R}), f' \in H^{s-1}(\mathbb{R}) \right\},$$

Theorem 1.2 ([7, Theorem 1.4., Proposition 5.1. and Theorem 5.4.]). *Let* $\lambda > 0$ *and* $s \geq 3$*. Suppose that* $(\rho_{in}, \phi_{in}) \in X^s(\mathbb{R}^d) \times \mathcal{C}(\mathbb{R})$ *with* $\phi'_{in} \in X^s(\mathbb{R})$ *and* $\rho(x) \geq \rho_{0*}$ *for some positive constant* ρ_{0*}*. Then there exist* $T > 0$ *independent of* $s \geq 3$ *and* $\varepsilon \in [0,1]$ *and a unique solution* $(a_\varepsilon, v_\varepsilon)$ *in* $\mathcal{C}([0,T]; X^s(\mathbb{R}) \times X^s(\mathbb{R}))$ *to* (1.4) *in dimension* $d = 1$*, i.e.:*

$$\begin{cases} \partial_t v_\varepsilon + \partial_x v_\varepsilon \, v_\varepsilon + \lambda \, \partial_x (\ln |a_\varepsilon|^2) = 0, & v_\varepsilon(0,x) = \phi'_{in}(x), \\ \partial_t a_\varepsilon + v_\varepsilon \cdot \partial a_\varepsilon + \dfrac{a_\varepsilon}{2} \partial_x v_\varepsilon = i \dfrac{\varepsilon}{2} \nabla a_\varepsilon, & a_\varepsilon(0,x) = a_{in}(x). \end{cases}$$

If we suppose in addition $\phi'_{in} \in L^2(\mathbb{R})$*, then for every* $\varepsilon > 0$, $u_\varepsilon = a_\varepsilon e^{i\frac{\phi_\varepsilon}{\varepsilon}}$ *(where* ϕ_ε *is given by* (1.5)*) is the unique solution in* $\mathcal{C}([0,T], X^s(\mathbb{R}))$ *to* (1.3) *(*$d = 1$*) with* $u_{\varepsilon,in} = \sqrt{\rho_{in}}\, e^{i\frac{\phi_{in}}{\varepsilon}}$.

Moreover, the Wigner Transform W_ε *of* u_ε *weakly converges in* $\mathcal{M}_b([0,T] \times \mathbb{R}^2)$ *to the bounded measure*

$$\mu(t, dx, d\xi) = |a(t,x)|^2 dx \otimes \delta_{\xi = v(t,x)}$$

where (v, a) *is the solution to* (1.4) *for* $\varepsilon = 0$*. Moreover,* μ *is a solution to* (1.1) *in* $d = 1$*:*

$$\partial\mu + \xi \, \partial_x \mu - \lambda \, \partial_x (\ln \rho) \, \partial_\xi \mu = 0, \qquad \text{where } \rho(t,x) = \int_{\mathbb{R}} \mu(t, x, d\xi).$$

This result shows that, far from the vacuum, the formal proof made previously actually holds. In particular, the link with the Isothermal Euler System still holds from the fact that the solutions are mono-kinetic, which is not the case in general. But, even if this result is enlightening, it will not be helpful in the case where the initial data is in a Sobolev space. Yet, we still have some other interesting results in the previous article ([7, Theorem 1.1.] and its proof) for a particular case in which we can compute explicitly the solutions for the Schrödinger and Euler equations: the Gaussian-monokinetic case.

Theorem 1.3 ([7, Theorem 1.1.] and its proof)**.** *Let λ, ρ_*, $\sigma_0 > 0$ and ω_0, $p_0 \in \mathbb{R}$. Set*

$$\rho_{in}(x) = \rho_* e^{-\sigma_0 x^2}, \qquad \phi_{in} = \omega_0 \frac{x^2}{2} + p_0 x, \qquad v_{in}(x) = \phi'_{in}(x),$$

and consider the solution τ_0 to the ordinary differential equation

$$\ddot{\tau}_0 = \frac{2\lambda\sigma_0}{\tau_0}, \qquad \tau_0(0) = 1, \qquad \dot{\tau}_0(0) = \omega_0.$$

Then $\tau_0 \in \mathcal{C}^\infty(\mathbb{R}^+)$. Set

$$\rho(t,x) = \frac{\rho_*}{\tau_0(t)} e^{-\sigma_0 \frac{(x-p_0t)^2}{\tau_0(t)^2}}, \qquad v(t,x) = \frac{\dot{\tau}_0(t)}{\tau_0(t)}(x - p_0 t) + p_0$$

and consider u_ε the solution of

$$\begin{cases} i\,\varepsilon\,\partial_t u_\varepsilon + \dfrac{\varepsilon^2}{2}\Delta u_\varepsilon = \lambda u_\varepsilon \ln |u_\varepsilon|^2, \\ u_\varepsilon(0,x) = u_{\varepsilon,in}(x) = \sqrt{\rho_{in}(x)}\, e^{i\,\frac{\phi_{in}(x)}{\varepsilon}} \in \mathcal{F}(H^1) \cap H^1(\mathbb{R}), \end{cases}$$

in $L^\infty(\mathbb{R}^+, \mathcal{F}(H^1) \cap H^1(\mathbb{R})) \cap \mathcal{C}(\mathbb{R}^+, L^2 \cap H^1_w(\mathbb{R}))$.

Then the Wigner Transform of u_ε weakly converges when $\varepsilon \to 0$ to the bounded measure

$$\mu(t, dx, d\xi) = \rho(t,x)dx \otimes \delta_{\xi = v(t,x)},$$

solution to (1.1) *with $\mu(0, dx, d\xi) = \rho_{in}(x)dx \otimes \delta_{\xi = v_{in}(x)}$ since (ρ, v) is solution to* (1.2)*.*

In particular, in the same theorem, it is shown that

$$\tau_0(t) \underset{t\to\infty}{\sim} 2\,t\sqrt{\lambda\sigma \ln t},$$

which yields

$$\frac{\tau_0(t)^2}{\sigma} \underset{t\to\infty}{\sim} 4\lambda\, t \ln t \underset{t\to\infty}{\sim} \tau(t)^2$$

where τ is defined in (1.10). Therefore, we still have for this case the same dispersion, and we see that, up to the same rescaling as before, ρ converges strongly to γ^2 in L^1. This result brings us to think that the behaviour found for the Schrödinger Equation still holds for the Vlasov Equation. This has already been proven for the case of general mono-kinetic solutions, *i.e.* for solutions of (1.2), in [4].

1.3 Main Results

The first idea was to generalize Theorem 1.1 to the case $\varepsilon > 0$ with some interesting initial data $u_{\varepsilon,in}$, after rescaling in a good way, in order to have some uniformity in ε in the estimates. Such results are presented in the following theorem:

Theorem 1.4. *Let $\lambda > 0$, $\rho_{in} \geq 0$ and ϕ_{in} such that:*

$$\sqrt{\rho_{in}} \in \mathcal{F}(H^1) \cap H^1(\mathbb{R}^d) \setminus \{0\}, \qquad \phi_{in} \in W^{1,1}_{loc}(\mathbb{R}^d), \qquad \sqrt{\rho_{in}}\, \nabla \phi_{in} \in L^2(\mathbb{R}^d). \tag{1.11}$$

For all $\varepsilon > 0$, there exists a unique global solution u_ε in $L^\infty(\mathbb{R}, \mathcal{F}(H^1) \cap H^1) \cap \mathcal{C}(\mathbb{R}, L^2 \cap H^1_w)$ to (1.3) *with $u_{\varepsilon,in} = \sqrt{\rho_{in}}\, e^{i\frac{\phi_{in}}{\varepsilon}}$:*

$$\begin{cases} i\,\varepsilon\, \partial_t u_\varepsilon + \dfrac{\varepsilon^2}{2} \Delta u_\varepsilon = \lambda u_\varepsilon \ln |u_\varepsilon|^2, \\ u_\varepsilon(0,x) = u_{\varepsilon,in}(x) = \sqrt{\rho_{in}(x)}\, e^{i\frac{\phi_{in}(x)}{\varepsilon}} \in \mathcal{F}(H^1) \cap H^1. \end{cases}$$

Rescale the solution

$$u_\varepsilon(t,x) = \frac{1}{\tau(t)^{\frac{d}{2}}} \frac{||\sqrt{\rho_{in}}||_{L^2}}{||\gamma||_{L^2}}\, v_\varepsilon\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot{\tau}(t)}{\tau(t)}\frac{|x|^2}{2\,\varepsilon}}, \tag{1.12}$$

where we recall τ is defined by (1.10) *and $\gamma(x) = e^{-\frac{|x|^2}{2}}$.*

There exists C independent of ε such that for all $t \geq 0$ and all $\varepsilon > 0$,

$$\int_{\mathbb{R}^d} \left(1 + |y|^2 + \left|\ln |v_\varepsilon(t,y)|^2\right|\right) |v_\varepsilon(t,y)|^2\, dy + \frac{\varepsilon^2}{\tau(t)^2} ||\nabla_y v_\varepsilon(t)||^2_{L^2(\mathbb{R}^d)} \leq C, \tag{1.13}$$

$$\int_0^\infty \frac{\varepsilon^2\, \dot{\tau}(t)}{\tau^3(t)} ||\nabla_y v_\varepsilon(t)||^2_{L^2(\mathbb{R}^d)} dt \leq C. \tag{1.14}$$

We have moreover three constants $C_1, C_2 \in \mathbb{R}^d$ and $C_3 > 0$ (which do not depend on t or on ε) such that for all $t > 1$ and all $\varepsilon > 0$,

$$\int_{\mathbb{R}^d} y\, |v_\varepsilon(t,y)|^2\, dy = \frac{1}{\tau(t)}(C_1 t + C_2), \tag{1.15}$$

$$\left| \int_{\mathbb{R}^d} |y|^2\, |v_\varepsilon(t,y)|^2\, dy - \int_{\mathbb{R}^d} |y|^2\, \gamma^2(y)\, dy \right| \leq C_3 \frac{\dot{\tau}(t) + 1}{\dot{\tau}(t)^2}, \tag{1.16}$$

which yields

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} |v_\varepsilon(t,y)|^2\, dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \gamma^2(y)\, dy \tag{1.17}$$

uniformly in ε. Finally,

$$|v_\varepsilon(t,.)|^2 \underset{t\to\infty}{\rightharpoonup} \gamma^2 \qquad \text{weakly in } L^1(\mathbb{R}^d).$$

Following the idea of the formal link with the Isothermal Euler System, such estimates would allow us to have the same kind of estimates for $\varepsilon = 0$, as soon as we make the link in a more rigorous way, which means, following the usual results, that the Wigner Transform of the solution of (1.3) converges to a measure solution to (1.1). However, our case is more tricky than the usual one. Indeed, the main point in Theorem 1.2 is that we can actually make (1.4) rigorous (in dimension 1) when we are far from the vacuum, and this is why we end up with a solution of the Vlasov equation. This is not the case in general for our approach where $u_{\varepsilon,in}$ is in a Sobolev space, and we need to take another way. Yet, the properties on the solution u_ε that the previous theorem induces are sufficient in order to have the convergence of the Wigner Transform and to characterize the behaviour of the limit. Defining the following space of test functions:

$$\mathcal{A} = \left\{ \phi \in \mathcal{C}_0(\mathbb{R}^d_x \times \mathbb{R}^d_\xi), (\mathcal{F}_\xi \phi)(x,z) \in L^1(\mathbb{R}^d_z, \mathcal{C}_0(\mathbb{R}^d_x)) \right\},$$

and the following spaces

$$L^1_2(\mathbb{R}^d) = \left\{ f \in L^1(\mathbb{R}^d), \int_{\mathbb{R}^d} |x|^2\, |f(x)|\, dx < \infty \right\},$$
$$L\log L(\mathbb{R}^d) = \left\{ f \in L^1(\mathbb{R}^d), \int_{\mathbb{R}^d} |f(x)||\ln|f(x)||\, dx < \infty \right\},$$

we can state the main result for this Wigner Measure.

Theorem 1.5. *Let $\lambda > 0$, $\rho_{in} \geq 0$ and ϕ_{in} satisfying* (1.11) *and, for all $\varepsilon > 0$, let u_ε be the solution in $L^\infty(\mathbb{R}^+, \mathcal{F}(H^1) \cap H^1) \cap \mathcal{C}(\mathbb{R}^+, L^2 \cap H^1_w)$ of* (1.3)*. Recall the definition of v_ε in* (1.12)*. Take W_ε (resp. $\tilde{W}_\varepsilon$) the Wigner Transform of u_ε (resp. v_ε). Then*

$$W_\varepsilon(0) \underset{n\to\infty}{\longrightarrow} \rho_{in}(x) \otimes \delta_{\xi=\nabla\phi_{in}(x)} \qquad \textit{in } \mathcal{A}', \tag{1.18}$$
$$\tilde{W}_\varepsilon(0) \underset{n\to\infty}{\longrightarrow} \frac{||\gamma^2||_{L^1}}{||\rho_{in}||_{L^1}} \rho_{in}(x) \otimes \delta_{\xi=\nabla\phi_{in}(x)} \qquad \textit{in } \mathcal{A}'. \tag{1.19}$$

Moreover there exists a subsequence $(\varepsilon_n)_n$ such that $\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$ and two (non-negative) finite measures W and $\tilde{W}$ in $L^\infty((0,\infty), \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d))$ such that for every $p \in [1,\infty)$

$$W_{\varepsilon_n} \underset{n\to\infty}{\rightharpoonup} W \qquad \textit{in } L^p_{loc}((0,\infty), \mathcal{A}'),$$
$$\tilde{W}_{\varepsilon_n} \underset{n\to\infty}{\rightharpoonup} \tilde{W} \qquad \textit{in } L^p_{loc}((0,\infty), \mathcal{A}'),$$

and the relation between W_ε and $\tilde{W}_\varepsilon$ given by

$$W_\varepsilon(t,x,\xi) = \frac{||\rho_{in}||_{L^1}}{||\gamma^2||_{L^1}}\, \tilde{W}_\varepsilon\left(t, \frac{x}{\tau(t)}, \tau(t)\,\xi - \dot{\tau}(t)\,x\right) \tag{1.20}$$

still holds after passing to the limit. Furthermore, we have

$$\tilde{W}(t, \mathbb{R}^d \times \mathbb{R}^d) = ||\gamma^2||_{L^1} \qquad \textit{for all } t \geq 0, \tag{1.21}$$
$$\tilde{\rho}(t,y) = \int_{\mathbb{R}^d} \tilde{W}(t,y,d\eta) \in L^\infty((0,\infty), L^1_2 \cap L\log L(\mathbb{R}^d)) \cap \mathcal{C}(\mathbb{R}^+, W^{-1,1} \cap L^1_w(\mathbb{R}^d)), \tag{1.22}$$

and there exist three constants $C_0 > 0$ and $C_1, C_2 \in \mathbb{R}^d$ such that for all $t \geq 0$,

$$\frac{1}{\tau(t)^2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\eta|^2\, \tilde{W}(t,dy,d\eta) \leq C_0,$$
$$\int_0^\infty \frac{\dot{\tau}(t_0)}{\tau^3(t_0)} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\eta|^2\, \tilde{W}(t_0,dy,d\eta)dt_0 \leq C_0,$$
$$\int_{\mathbb{R}^d} y\, \tilde{\rho}(t,y)\, dy = \frac{1}{\tau(t)}(C_1 t + C_2),$$

which yield

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \end{pmatrix} \tilde{\rho}(t,y)\, dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \end{pmatrix} \gamma^2(y)\, dy.$$

Finally,

$$\tilde{\rho}(t,.) \underset{t\to\infty}{\rightharpoonup} \gamma^2 \qquad \textit{weakly in } L^1(\mathbb{R}^d).$$

Therefore, the main behaviour of the solution of the Logarithmic Schrödinger Equation in Theorem 1.1 is propagated to the Wigner Measure, thanks to the uniformity of the bounds found in Theorem 1.4. However, we do not prove the fact that the Wigner Measure satisfies (1.1). Indeed, to the best of our knowledge, the results to prove it rigorously in the general case ask for a little bit of regularity for the potential V, in both linear and non-linear cases, whereas our potential $\ln|u_\varepsilon|^2$ is highly singular and non-linear.

Remark 1.1. The convergence for the second momentum stated in (1.16) is uniform in ε. Yet, we still can not conclude for the case "$\varepsilon = 0$" because we do not know if $\int_{\mathbb{R}^d} |y|^2\, \tilde{\rho}_\varepsilon(t,y)\, dy$ converges to $\int_{\mathbb{R}^d} |y|^2\, \tilde{\rho}(t,y)\, dy$. This would have been the case if, for example, we had a bound of a higher moment, but we do not.

Remark 1.2.

- As a straightforward consequence, with the previous notations of Theorem 1.4, we infer the slightly weaker property that $|v_\varepsilon(t)|^2$ converges to γ^2 in Wasserstein distance:

$$W_2\left(\frac{|v_\varepsilon(t)|^2}{\pi^{\frac{d}{2}}}, \frac{\gamma^2}{\pi^{\frac{d}{2}}}\right) \underset{t\to\infty}{\longrightarrow} 0,$$

where we recall that the Wasserstein distance is defined, for ν_1 and ν_2 probability measures, by

$$W_p(\nu_1,\nu_2) = \inf\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p\, d\mu(x,y)\right)^{\frac{1}{p}}; \quad (\pi_j)_\#\mu = \nu_j\right\},$$

where μ varies among all probability measures on $\mathbb{R}^d \times \mathbb{R}^d$, and $\pi_j : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ denotes the canonical projection onto the j-th factor (see e.g. [13]).

- In the same way, with the notations of Theorem 1.5, thanks to the bound uniform in t for $|y|^2\, \tilde{\rho}(t)$ in L^1 and the weak convergence of $\tilde{\rho}(t)$ as $t\to\infty$, we infer that for every $p \in [1,2)$, $\int_{\mathbb{R}^d} |y|^p\, \tilde{\rho}(t)\, dy$ converges to $\int_{\mathbb{R}^d} |y|^p\, \gamma^2\, dy$ (see Appendix B for a proof). Therefore, $\tilde{\rho}(t)$ converges to γ^2 in Wasserstein distance:

$$W_p\left(\frac{\tilde{\rho}(t)}{\pi^{\frac{d}{2}}}, \frac{\gamma^2}{\pi^{\frac{d}{2}}}\right) \underset{t\to\infty}{\longrightarrow} 0,$$

for every $p \in [1,2)$. This remark shows that we might have some uniformity in $\varepsilon \in [0,1]$ for the convergence of $|v_\varepsilon(t)|^2$ to γ^2 in Wasserstein distance W_p for every $p \in [1,2)$.

We are now interested in the behaviour of the solutions of the Logarithmic Vlasov Equation. In the light of the previous theorem and of Theorem 1.3, our intuition would say that any solution of (1.1) has the same behaviour as previously. However, the potential is actually so much singular that the way to formalize (1.1) is very difficult, and therefore getting solutions is even more difficult in the general case. Yet, we can still compute some explicit solutions, like in Theorem 1.3 for the case of Gaussian-monokinetic initial data. We also have another class of explicit solutions, which is in some way an extension of the previous case: the "Gaussian-Gaussian" case.

Theorem 1.6. *1. For $c_{1,0} > 0$, $c_{2,0} > 0$ and $c_{1,1}, B_0, B_1 \in \mathbb{R}$, set*

$$\tilde{C} := c_{1,0}\, c_{2,0}, \tag{1.23}$$

and consider $c_1 \in \mathcal{C}^\infty(\mathbb{R}^+)$ the solution of the ordinary differential equation

$$\begin{cases} \ddot{c}_1 = \dfrac{2\lambda}{c_1} + \dfrac{\tilde{C}^2}{c_1^3}, & (1.24)\\ c_1(0) = c_{1,0}, & (1.25)\\ \dot{c}_1(0) = c_{1,1}. & (1.26) \end{cases}$$

Then, set

$$c_2(t) := \frac{\tilde{C}}{c_1(t)}, \tag{1.27}$$

$$b_1(t) := B_1 t + B_0, \tag{1.28}$$

$$b_2(t,x) := \frac{\dot{c}_1(t)}{c_1(t)}(x - B_1\, t - B_0) + B_1. \tag{1.29}$$

The function

$$f(t,x,\xi) = \frac{1}{\pi\, c_1(t)\, c_2(t)}\ \exp\left[-\frac{|x-b_1(t)|^2}{c_1(t)^2} - \frac{|\xi - b_2(t,x)|^2}{c_2(t)^2}\right]$$

satisfies (1.1)*. Moreover, if we rescale to $\tilde{f} = \tilde{f}(t,y,\eta)$ like previously*

$$f(t,x,\xi) := \frac{1}{||\gamma^2||_{L^1}}\ \tilde{f}\left(t, \frac{x}{\tau(t)}, \tau(t)\,\xi - \dot{\tau}(t)\,x\right),$$

and define

$$\tilde{\rho}(t,y) := \int_{\mathbb{R}^d} f(t,y,\eta)\,d\eta,$$

there holds

$$\tilde{\rho}(t,.) \underset{t\to\infty}{\longrightarrow} \gamma^2 \qquad \textit{strongly in } L^1(\mathbb{R}).$$

2. *Let $T \in (0,+\infty]$, $b_1 = b_1(t) \in \mathcal{C}^1([0,T),\mathbb{R})$, $c_1 = c_1(t) \in \mathcal{C}^1([0,T),(0,\infty))$, $b_2 = b_2(t,x) \in \mathcal{C}^1([0,T)\times\mathbb{R},\mathbb{R})$ and $c_2 = c_2(t,x) \in \mathcal{C}^1([0,T)\times\mathbb{R},(0,\infty))$ such that*

$$f(t,x,\xi) = \frac{1}{(\pi\, c_1(t)\, c_2(t,x))^d}\ \exp\left[-\frac{|x-b_1(t)|^2}{c_1(t)^2} - \frac{|\xi - b_2(t,x)|^2}{c_2(t,x)^2}\right]$$

is a solution of (1.1)*. Then c_2 does not depend on x, all the functions are $\mathcal{C}^2$ and* (1.23)-(1.29) *hold.*

We see that the behaviour still holds, with the better result of strong convergence in L^1 to γ^2 for $\tilde{\rho}$, like what we had in Theorem 1.3.

However, we are not interested in the formalization of the equation (1.1) for the general case in this paper, and we focus only on the results by assuming some formal properties to be verified by such a "solution". The first properties that come to mind are the usual properties for Vlasov equations: the mass conservation and the energy conservation. Following the results found for the Wigner Measure, we also want equations which are similar to the isothermal Euler system, in order to proceed like the proofs of the above results, which are equations involving $\partial_t\rho$ and $\partial_t \int \xi f(t,x,d\xi)$, but also those involving some second momentum, in x for instance. Such remarks lead us to define:

$$\mathcal{M}\Sigma_{\log} = \left\{\mu \in \mathcal{M}(\mathbb{R}^d_x \times \mathbb{R}^d_\xi), \rho(x) = \int_{\mathbb{R}^d_\xi} \mu(x,d\xi) \in L^1(\mathbb{R}^d) \cap L\,\log L\,(\mathbb{R}^d)\right\},$$

$$\mathcal{M}_2 = \left\{\mu \in \mathcal{M}(\mathbb{R}^d_x \times \mathbb{R}^d_\xi), \iint_{\mathbb{R}^d\times\mathbb{R}^d} (|x|^2 + |\xi|^2)\,d\mu < \infty\right\},$$

where $\mathcal{M}$ stands for the space of non-negative measure. This yields the following theorem:

Theorem 1.7. *Let* $f = f(t,x,\xi) \in L^\infty_{loc}(0,\infty;\mathcal{M}\Sigma_{\log}\cap\mathcal{M}_2)$ *satisfying* $f(0,x,\xi) = f_{in}(x,\xi) \in \mathcal{M}\Sigma_{\log}\cap\mathcal{M}_2\setminus\{0\}$ *and*

$$\frac{d}{dt}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} f(t,dx,d\xi)\right) = 0, \tag{1.30}$$

$$\frac{d}{dt}\left(\frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} |\xi|^2 f(t,dx,d\xi) + \lambda\int_{\mathbb{R}^d}\rho(t,x)\ln\rho(t,x)dx\right) = 0, \tag{1.31}$$

$$\partial_t\rho(t,x) + \nabla_x\cdot\left(\int_{\mathbb{R}^d}\xi\, f(t,x,d\xi)\right) = 0, \tag{1.32}$$

$$\partial_t\int_{\mathbb{R}^d}\xi\, f(t,x,d\xi) + \nabla_x\cdot\int_{\mathbb{R}^d}\xi\otimes\xi\, f(t,x,d\xi) + \lambda\,\nabla_x\rho(t,x) = 0, \tag{1.33}$$

$$\frac{d}{dt}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}|x|^2 f(t,dx,d\xi)\right) = 2\iint_{\mathbb{R}^d\times\mathbb{R}^d} x\cdot\xi\, f(t,dx,d\xi), \tag{1.34}$$

$$\frac{d}{dt}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} x\cdot\xi\, f(t,dx,d\xi)\right) = \iint_{\mathbb{R}^d\times\mathbb{R}^d}|\xi|^2 f(t,dx,d\xi) + \lambda\int_{\mathbb{R}^d}\rho(t,x)dx. \tag{1.35}$$

Then $\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,d\xi) \in \mathcal{C}(\mathbb{R}^+, L^1_w(\mathbb{R}^d))$*. Rescale the solution to* $\tilde f = \tilde f(t,y,\eta)$ *by setting:*

$$f(t,x,\xi) = \frac{M}{||\gamma^2||_{L_1}}\,\tilde f\left(t,\frac{x}{\tau(t)},\tau(t)\xi - \dot\tau(t)x\right), \tag{1.36}$$

where $M = f_{in}(\mathbb{R}^d\times\mathbb{R}^d)$*, which leads to*

$$\rho(t,x) = \frac{M}{\tau^d(t)\,||\gamma^2||_{L_1}}\,\tilde\rho\left(t,\frac{x}{\tau(t)}\right),$$

where $\tilde\rho(t,y) = \int_{\mathbb{R}^d}\tilde f(t,y,d\eta)$*. There exists* $C>0$ *such that for all* $t\geq 0$*,*

$$\left(\int_{\mathbf{R}^d}\left(|\ln\tilde\rho(t,y)\,| + |y|^2\right)\tilde\rho(t,y)\,dy + \frac{1}{\tau^2(t)}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\eta|^2\tilde f(t,dy,d\eta)\right) \leq C. \tag{1.37}$$

We have moreover

$$\int_{\mathbb{R}^d}\begin{pmatrix}1\\ y\\ |y|^2\end{pmatrix}\tilde\rho(t,y)\,dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d}\begin{pmatrix}1\\ y\\ |y|^2\end{pmatrix}\gamma^2(y)\,dy.$$

Finally,

$$\tilde\rho(t,.) \underset{t\to\infty}{\rightharpoonup} \gamma^2 \qquad \text{weakly in } L^1(\mathbb{R}^d).$$

The assumptions (1.34) and (1.35) should come from (1.32) and (1.33). However, we still add those equations in the assumptions to make the theorem completely rigorous.

Remark 1.3. The result $\rho\in\mathcal{C}(\mathbb{R}^+, L^1_w(\mathbb{R}^d))$ is important in order to ensure that $\rho(t)$ is actually well-defined for *all* $t>0$ (and not only almost everywhere), despite the weaker regularity of f. This result actually comes from the assumption (1.32) along with $\int_{\mathbb{R}^d}|\xi|\, f(t,x,d\xi)\in L^\infty_{loc}(0,\infty;\mathcal{M}(\mathbb{R}^d))$.

The previous theorem predicts the behaviour of most of the involved expressions, it only remains $\iint_{\mathbb{R}^d_x\times\mathbb{R}^d_\xi}|\xi|^2 f(t,dx,d\xi)$. This is computed in the following corollary.

Corollary 1.1. *Let* f *satisfying the hypothesis of Theorem 1.7. Then it also satisfies*

$$\iint_{\mathbb{R}^d_x\times\mathbb{R}^d_\xi}|\xi|^2 f(t,dx,d\xi) \underset{t\to\infty}{\sim} 2\lambda dM\,\ln t.$$

Remark 1.4. In the same way as before, $\tilde\rho$ converges to γ^2 in Wasserstein distance:

$$W_2\left(\frac{\tilde\rho(t)}{\pi^{\frac{d}{2}}},\frac{\gamma^2}{\pi^{\frac{d}{2}}}\right)\underset{t\to\infty}{\longrightarrow} 0.$$

1.4 Plan of the proofs

The plan of the rest of the paper is the following. In Section 2.1, we recall how we get the existence and uniqueness of a solution to (1.3) with more general $u_{\varepsilon,in}$, and then we prove Theorem 1.4. In Section 2.2, we first define the Wigner Transform and state some general properties. Then, we prove Theorem 1.5. Last, Section 2.3 is devoted to the proofs of Theorems 1.6 and 1.7 and Corollary 1.1.

Chapter 2

Results and proofs

2.1 Schrödinger Equation with logarithmic non-linearity

We are interested in the properties of the solution of the Schrödinger Equation with logarithmic non-linearity:

$$i\varepsilon\, \partial_t u_\varepsilon + \frac{\varepsilon^2}{2}\Delta u_\varepsilon = \lambda\, u_\varepsilon \ln |u_\varepsilon|^2, \qquad \text{in } \mathcal{D}', \qquad\qquad u_\varepsilon(0,x) = u_{\varepsilon,in}(x),$$

with $\lambda > 0$, $\varepsilon > 0$ and $u_{\varepsilon,in}$ given. First, we will recall the existence and uniqueness of such a solution when the initial data $u_{\varepsilon,in}$ has some good properties before going into the proof of Theorem 1.4.

2.1.1 Existence and uniqueness of a solution

First, recall the Logarithmic Schrödinger Equation (1.9) like above with $\varepsilon = 1$:

$$i\, \partial_t u + \frac{1}{2}\Delta u = \lambda\, u \ln |u|^2, \qquad\qquad u(0,x) = u_{in}(x),$$

with $\lambda > 0$ and u_{in} given. This equation has already been studied in [5] and [8], and we have some results for existence and uniqueness. We recall before the mass, angular momentum and energy for some $f \in \left\{g \in H^1(\mathbb{R}^d), |g|^2 \ln |g|^2 \in L^1\right\}$:

$$\begin{aligned} M(f) &:= ||f||^2_{L^2(\mathbb{R}^d)}, \\ J(f) &:= \operatorname{Im} \int_{\mathbb{R}^d} \overline{f(x)} \nabla f(x)\, dx, \\ E(f) &:= \frac{1}{2}||\nabla f||^2_{L^2(\mathbb{R}^d)} + \lambda \int_{\mathbb{R}^d} |f(x)|^2 \ln |f(x)|^2\, dx. \end{aligned}$$

Such definitions allow us to correctly state the following theorem:

Theorem 2.1 ([5, Theorem 1.5.]). *Let the initial data u_{in} belong to $\mathcal{F}(H^1) \cap H^1(\mathbb{R}^d)$. There exists a unique, global solution $u \in L^\infty_{loc}(\mathbb{R}; \mathcal{F}(H^1) \cap H^1(\mathbb{R}^d)) \cap \mathcal{C}(\mathbb{R}, L^2 \cap H^1_w(\mathbb{R}^d))$ to (1.9). Moreover, the mass $M(u(t))$, the angular momentum $J(u(t))$ and the energy $E(u(t))$ are independent of time.*

Remark 2.1. Weaker hypothesis are possible in order to have existence and uniqueness, in particular substituting $\mathcal{F}(H^1)$ by $\mathcal{F}(H^\alpha)$ for $\alpha \in (0, 1]$ (see [5]). However, this hypothesis is not helpful in the following, and we stick to the assumption $u_{in} \in \mathcal{F}(H^1) \cap H^1(\mathbb{R}^d)$.

We see that (1.3) and (1.9) are actually linked through a simple change of variables: if u_ε satisfies (1.3), then $w_\varepsilon(t,x) = u_\varepsilon(\varepsilon t, \varepsilon x)$ satisfies (1.9) with $w_\varepsilon(0,x) = u_\varepsilon(\varepsilon x) \in \mathcal{F}(H^1) \cap H^1(\mathbb{R}^d)$ if $u_{\varepsilon,in} \in \mathcal{F}(H^1) \cap H^1(\mathbb{R}^d)$. Such remark allows us to state the existence and uniqueness for the equation (1.3) and some first properties, after modification of the angular momentum and energy:

$$\begin{aligned} J_\varepsilon(f) &:= \operatorname{Im} \varepsilon \int_{\mathbb{R}^d} \overline{f}(x) \nabla f(x)\, dx, \\ E_\varepsilon(f) &:= \frac{\varepsilon^2}{2} ||\nabla f||^2_{L^2(\mathbb{R}^d)} + \lambda \int_{\mathbb{R}^d} |f(x)|^2 \ln |f(x)|^2 \, dx. \end{aligned}$$

Corollary 2.1. *Let $\varepsilon > 0$ and $u_{\varepsilon,in}$ belong to $\mathcal{F}(H^1) \cap H^1(\mathbb{R}^d)$. There exists a unique, global solution $u_\varepsilon \in L^\infty_{loc}(\mathbb{R}; \mathcal{F}(H^1) \cap H^1(\mathbb{R}^d)) \cap \mathcal{C}(\mathbb{R}, L^2 \cap H^1_w(\mathbb{R}^d))$ to (1.9). Moreover, the mass $M(u_\varepsilon(t))$, the angular momentum $J_\varepsilon(u_\varepsilon(t))$ and the energy $E_\varepsilon(u_\varepsilon(t))$ are independent of time.*

2.1.2 Properties of the solution

In view of Theorem 1.4, we now fix $u_{\varepsilon,in}$ like in the hypothesis of the theorem:

$$u_{\varepsilon,in}(x) = \sqrt{\rho_{in}(x)}\, e^{i \frac{\phi_{in}(x)}{\varepsilon}},$$

where $\rho_{in} \geq 0$ and ϕ_{in} are such that

$$\sqrt{\rho_{in}} \in \mathcal{F}(H^1) \cap H^1(\mathbb{R}^d) \setminus \{0\}, \qquad \phi_{in} \in W^{1,1}_{\text{loc}}(\mathbb{R}^d), \qquad \sqrt{\rho_{in}}\, \nabla \phi_{in} \in L^2(\mathbb{R}^d),$$

so that for all $\varepsilon > 0$,

$$u_{\varepsilon,in} \in \mathcal{F}(H^1) \cap H^1(\mathbb{R}^d),$$

which yields a solution $u_\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}; \mathcal{F}(H^1) \cap H^1(\mathbb{R}^d)) \cap \mathcal{C}(\mathbb{R}, L^2 \cap H^1_w(\mathbb{R}^d))$ to (1.3) thanks to the previous corollary.

Rescaling the solution

Following the ideas of Theorem 1.1, we want to rescale the solutions in order to find similar properties on this rescaling. As we are aware that the main oscillations for (1.3) should be of order $\frac{1}{\varepsilon}$, especially since the oscillations of $u_{\varepsilon,in}$ are of the same order, we can guess that the good rescaling is (1.12), the one we introduced in Theorem 1.4, recalled here:

$$u_\varepsilon(t,x) = \frac{1}{\tau(t)^{\frac{d}{2}}} \frac{||\sqrt{\rho_{in}}||_{L^2}}{||\gamma||_{L^2}}\, v_\varepsilon\left(t, \frac{x}{\tau(t)}\right) e^{i \frac{\dot{\tau}(t)}{\tau(t)} \frac{|x|^2}{2\,\varepsilon}},$$

where $\gamma(x) = e^{-\frac{|x|^2}{2}}$ and τ is the solution of (1.10):

$$\ddot{\tau} = \frac{2\lambda}{\tau}, \qquad \tau(0) = 1, \qquad \dot{\tau}(0) = 0,$$

so that we have, when $t \to \infty$ (see [5, Lemma 1.6]),

$$\tau(t) = 2t\sqrt{\lambda \ln t}\left(1 + \mathcal{O}\left(\frac{\ln \ln t}{\ln t}\right)\right).$$

Writing (1.3) in terms of v_ε yields

$$i\varepsilon\, \partial_t v_\varepsilon + \frac{\varepsilon^2}{2\tau(t)^2} \Delta_y v_\varepsilon = \lambda v_\varepsilon \ln\left|\frac{v_\varepsilon}{\gamma}\right|^2 - \lambda\left(d \ln \tau(t) - 2\, \ln \frac{||\sqrt{\rho_{in}}||_{L^2}}{||\gamma||_{L^2}}\right) v_\varepsilon.$$

The last term is totally harmless, as it can be removed by changing v_ε into $v_\varepsilon\, e^{-i\frac{\theta}{\varepsilon}}$ where

$$\theta = \theta(t) = \lambda d \int_0^t \ln \tau(s)\, ds - 2\lambda t \ln \frac{||\sqrt{\rho_{in}}||_{L^2}}{||\gamma||_{L^2}}.$$

Thus, we obtain the system

$$i\,\varepsilon\,\partial_t v_\varepsilon + \frac{\varepsilon^2}{2\tau(t)^2}\Delta_y v_\varepsilon = \lambda v_\varepsilon \ln\left|\frac{v_\varepsilon}{\gamma}\right|^2, \qquad v_\varepsilon(0,x) = \frac{||\gamma||_{L^2}}{||\sqrt{\rho_{in}}||_{L^2}}\, u_{\varepsilon,in}.$$

Estimates

We now define the (modified) energy:

$$\mathcal{E}^\varepsilon(t) = \operatorname{Im}\int_{\mathbb{R}^d} v_\varepsilon(t,y)\,\varepsilon\,\overline{\partial_t v_\varepsilon}(t,y)\,dy = \mathcal{E}^\varepsilon_{\text{kin}}(t) + \lambda\,\mathcal{E}^\varepsilon_{\text{ent}}(t),$$

where

$$\mathcal{E}^\varepsilon_{\text{kin}}(t) = \frac{\varepsilon^2}{2\,\tau(t)^2}||\nabla v_\varepsilon||^2_{L^2},$$
$$\mathcal{E}^\varepsilon_{\text{ent}}(t) = \int_{\mathbb{R}^d} |v_\varepsilon(t,y)|^2 \ln\left|\frac{v_\varepsilon(t,y)}{\gamma(y)}\right|^2,$$

are respectively the modified kinetic and entropy energies.

Then we easily compute:

$$\dot{\mathcal{E}^\varepsilon} = -2\frac{\dot\tau(t)}{\tau(t)}\mathcal{E}^\varepsilon_{\text{kin}}. \tag{2.1}$$

Following the ideas of [5], we should now have bounds independent of ε if $\mathcal{E}^\varepsilon(0)$ is bounded independently of ε. Thanks to the assumptions (1.11), we can compute explicitly $\mathcal{E}^\varepsilon(0)$ in terms of ρ_{in} and ϕ_{in}:

$$\begin{aligned}
\mathcal{E}^\varepsilon(0) &= \frac{\varepsilon^2}{2}\frac{||\gamma^2||_{L^1}}{||\sqrt{\rho_{in}}||^2_{L^2}}||\nabla u_{\varepsilon,in}||^2_{L^2} + \lambda\frac{||\gamma^2||_{L^1}}{||\sqrt{\rho_{in}}||^2_{L^2}}\ln\frac{||\gamma^2||_{L^1}}{||\sqrt{\rho_{in}}||^2_{L^2}}\int |u_{\varepsilon,in}|^2\,dx \\
&\qquad + \lambda\frac{||\gamma^2||_{L^1}}{||\sqrt{\rho_{in}}||^2_{L^2}}\int |u_{\varepsilon,in}|^2 \ln|u_{\varepsilon,in}|^2\,dx + \lambda\frac{||\gamma^2||_{L^1}}{||\sqrt{\rho_{in}}||^2_{L^2}}\int |u_{\varepsilon,in}|^2\,|y|^2\,dx \\
&= \frac{\varepsilon^2}{2}\frac{||\gamma^2||_{L^1}}{||\sqrt{\rho_{in}}||^2_{L^2}}\left|\left|\nabla\left(\sqrt{\rho_{in}}\right) + i\frac{\nabla\phi_{in}}{\varepsilon}\sqrt{\rho_{in}}\right|\right|^2_{L^2} + \lambda\frac{||\gamma^2||_{L^1}}{||\sqrt{\rho_{in}}||^2_{L^2}}\ln\frac{||\gamma^2||_{L^1}}{||\sqrt{\rho_{in}}||^2_{L^2}}||\sqrt{\rho_{in}}||^2_{L^2} \\
&\qquad + \lambda\frac{||\gamma^2||_{L^1}}{||\sqrt{\rho_{in}}||^2_{L^2}}\int \rho_{in}\ln\rho_{in}\,dx + \lambda\frac{||\gamma^2||_{L^1}}{||\sqrt{\rho_{in}}||^2_{L^2}}\int \rho_{in}\,|y|^2\,dx \\
&= \frac{||\gamma^2||_{L^1}}{2\,||\sqrt{\rho_{in}}||^2_{L^2}}\,||\varepsilon\,\nabla\left(\sqrt{\rho_{in}}\right) + i\nabla\phi_{in}\sqrt{\rho_{in}}||^2_{L^2} + \lambda\ln\frac{||\gamma^2||_{L^1}}{||\sqrt{\rho_{in}}||^2_{L^2}}||\gamma^2||_{L^1} \\
&\qquad + \lambda\frac{||\gamma^2||_{L^1}}{||\sqrt{\rho_{in}}||^2_{L^2}}\int \rho_{in}\ln\rho_{in}\,dx + \lambda\frac{||\gamma^2||_{L^1}}{||\sqrt{\rho_{in}}||^2_{L^2}}\int \rho_{in}\,|y|^2\,dx
\end{aligned}$$

The hypothesis (1.11) actually yields $\int \rho_{in}\,|\ln\rho_{in}\,|\,dy < \infty$. Indeed, $\rho_{in} \geq 0$, and

$$\rho_{in}\,|\ln\rho_{in}\,| \leq C_\delta\,(\rho_{in}^{1-\delta} + \rho^{1+\delta})$$

for all $\delta \in [0,1)$. Then, the two bounds

$$\int \rho_{in}^{1+\delta} \leq C_\delta ||\sqrt{\rho_{in}}||^{1+\delta}_{H^1}$$

for $\delta > 0$ small enough thanks to Sobolev embeddings and

$$\int \rho_{in}^{1-\delta} \leq C_\delta ||\sqrt{\rho_{in}}||_{L^2}^{2-2\delta-d\delta} |||x|\sqrt{\rho_{in}}||_{L^2}^{d\delta}$$

for $0 < \delta < \frac{2}{d+2}$ yield $\int \rho_{in} \,|\ln \rho_{in}\,|\, dy < \infty$.

Moreover, $\nabla\left(\sqrt{\rho_{in}}\right) \in L^2(\mathbb{R}^d)$ and all the other terms are well-defined thanks to (1.11). Therefore we have a bound independent of ε as previously announced. Thus, this leads to the result, which are the estimates (1.13) and (1.14) of Theorem 1.4 stated in the following lemma:

Lemma 2.1. *Under the assumption* (1.11)*, we have*

$$\sup_{\substack{t\geq 0\\ \varepsilon>0}} \left(\int_{\mathbf{R}^d} \left(|\ln |v_\varepsilon|^2(t,y)\,| + |y|^2\right) |v_\varepsilon|^2(t,y)\, dy + \frac{\varepsilon^2}{2\tau^2(t)} \int_{\mathbb{R}^d} |\nabla v_\varepsilon|^2(t,y)\, dy \right) < \infty,$$

and

$$\int_0^\infty \frac{\varepsilon^2\, \dot{\tau}(t)}{\tau^3(t)} \int_{\mathbb{R}^d} |\nabla v_\varepsilon|^2(t,y)\, dydt \leq C.$$

Proof. Write

$$\mathcal{E}_{\text{ent}}^\varepsilon = \int |v_\varepsilon|^2 \ln |v_\varepsilon|^2 + \int |y|^2\, |v_\varepsilon|^2,$$

and

$$\int |v_\varepsilon| \ln |v_\varepsilon|^2 = \int_{|v_\varepsilon|>1} |v_\varepsilon|^2 \ln |v_\varepsilon|^2 + \int_{|v_\varepsilon|\leq 1} |v_\varepsilon|^2 \ln |v_\varepsilon|^2.$$

Then, separating the positive and negative parts of the modified energy and using the fact that this modified energy is non-increasing, we have

$$\mathcal{E}_+^\varepsilon := \mathcal{E}_{\text{kin}}^\varepsilon + \lambda \int_{|v_\varepsilon|>1} |v_\varepsilon|^2 \ln |v_\varepsilon|^2 + \lambda \int |y|^2\, |v_\varepsilon|^2 \leq \mathcal{E}^\varepsilon(0) - \lambda \int_{|v_\varepsilon|\leq 1} |v_\varepsilon| \ln |v_\varepsilon|^2 \leq C + \mathcal{E}_-^\varepsilon,$$

where $\mathcal{E}_-^\varepsilon = -\lambda \int_{|v_\varepsilon|\leq 1} |v_\varepsilon| \ln |v_\varepsilon|^2 \geq 0$. This term is controlled by

$$\mathcal{E}_-^\varepsilon \leq C_\delta \int_{\mathbb{R}^d} |v_\varepsilon|^{2-\delta},$$

for all $\delta \in (0,2)$. Moreover, we have the estimate

$$\int_{\mathbb{R}^d} |v_\varepsilon|^{2-\delta} \leq C_\delta\, ||v_\varepsilon||_{L^2}^{2-\left(1+\frac{d}{2}\right)\delta} ||yv_\varepsilon||_{L^2}^{\frac{d\delta}{2}} = C_\delta\, ||\gamma||_{L^2}^{2-\left(1+\frac{d}{2}\right)\delta} ||yv_\varepsilon||_{L^2}^{\frac{d\delta}{2}},$$

as soon as $0 < \delta < \frac{2}{d+2}$. Taking (for example) $\delta = \frac{1}{d+2}$, this implies

$$\mathcal{E}_-^\varepsilon \leq C_d \left(\mathcal{E}_+^\varepsilon\right)^{\frac{d}{4(d+2)}}$$

$$\mathcal{E}_+^\varepsilon \leq C_d \left(1 + \left(\mathcal{E}_+^\varepsilon\right)^{\frac{d}{4(d+2)}}\right),$$

and thus $\mathcal{E}_+^\varepsilon \leq \tilde{C}$ with $\tilde{C}$ which does not depend on t and ε since $\frac{d}{4(d+2)} < 1$. Then we also have $\mathcal{E}_-^\varepsilon \leq C_0$, and it follows that $\mathcal{E}_+^\varepsilon + \mathcal{E}_-^\varepsilon$ and $\mathcal{E}^\varepsilon$ are bounded uniformly in $t \geq 0$ and in $\varepsilon > 0$.

Last, (1.14) follows from (2.1) and the fact that $\mathcal{E}^\varepsilon(t)$ is bounded uniformly in $\varepsilon > 0$ and in $t \geq 0$. □

Remark 2.2. Actually, we have $\mathcal{E}^\varepsilon \geq \mathcal{E}_{\text{ent}}^\varepsilon \geq 0$ thanks to the Csiszár-Kullback inequality, which reads (see [1, Theorem 8.2.7])

$$\mathcal{E}_{\text{ent}}^\varepsilon(t) \geq \frac{1}{2||\gamma^2||_{L^1(\mathbb{R}^d)}} \left|\left|\, |v_\varepsilon|^2(t) - \gamma^2 \right|\right|_{L^1(\mathbb{R}^d)}^2.$$

This inequality shows that, if we had $\mathcal{E}_{\text{ent}}^\varepsilon(t) \underset{t\to\infty}{\longrightarrow} 0$ (for example, $\mathcal{E}^\varepsilon(t) \underset{t\to\infty}{\longrightarrow} 0$), then we would have $\left|\left|\, |v_\varepsilon|^2(t) - \gamma^2 \right|\right|_{L^1(\mathbb{R}^d)}^2 \underset{t\to\infty}{\longrightarrow} 0$ and then strong convergence would follow, but we cannot reach this conclusion in the general case.

Convergence of some quadratic quantities

We now prove (1.15)-(1.17), as stated in the next lemma.

Lemma 2.2. *Under the assumptions of Theorem 1.4, there holds for some constants $C_1, C_2 \in \mathbb{R}^d$ and $C_3 > 0$ (which do not depend on t and on ε) and for all $t > 1$ and all $\varepsilon > 0$,*

$$\int_{\mathbb{R}^d} y\, |v_\varepsilon(t,y)|^2\, dy = \frac{1}{\tau(t)}(C_1 t + C_2),$$
$$\left| \int_{\mathbb{R}^d} |y|^2\, |v_\varepsilon(t,y)|^2\, dy - \int_{\mathbb{R}^d} |y|^2\, \gamma^2(y)\, dy \right| \leq C_3 \frac{\dot\tau(t)+1}{\dot\tau(t)^2},$$

which yields

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} |v_\varepsilon(t,y)|^2\, dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \gamma^2(y)\, dy$$

uniformly in ε.

Proof. For the first equation, following the ideas of [5, Lemma 4.2], we introduce

$$I_1^\varepsilon(t) = \operatorname{Im} \varepsilon \int_{\mathbb{R}^d} \overline{v_\varepsilon}(t,y)\, \nabla v_\varepsilon(t,y)\, dy, \qquad I_2^\varepsilon(t) = \int_{\mathbb{R}^d} y\, |v_\varepsilon(t,y)|^2\, dy,$$

and we compute

$$\dot{I_1^\varepsilon} = -2\lambda I_2^\varepsilon, \qquad \dot{I_2^\varepsilon} = \frac{1}{\tau^2(t)} I_1^\varepsilon.$$

Set $\tilde{I_2^\varepsilon} = \tau\, I_2^\varepsilon$: we have $\ddot{\tilde{I_2^\varepsilon}} = 0$ and therefore $I_2^\varepsilon(t) = \frac{1}{\tau(t)}(-I_1^\varepsilon(0)\, t + I_2^\varepsilon(0))$. Then, computing the two terms in the right-hand side shows that they do not depend on ε:

$$I_1^\varepsilon(0) = \frac{||\sqrt{\rho_{in}}||_{L^2}}{||\gamma||_{L^2}} \int_{\mathbb{R}^d} \rho_{in}(x)\, \nabla\phi_{in}(x)\, dx,$$
$$I_2^\varepsilon(0) = \frac{||\sqrt{\rho_{in}}||_{L^2}}{||\gamma||_{L^2}} \int_{\mathbb{R}^d} x\, \rho_{in}(x)\, dx,$$

which leads to the first equality. We now go back to the conservation of energy for u_ε,

$$\frac{\varepsilon^2}{2}||\nabla u_\varepsilon||_{L^2}^2 + \lambda \int_{\mathbb{R}^d} |u_\varepsilon(t,x)|^2 \ln |u_\varepsilon(t,x)|^2\, dx = \frac{1}{2}\, ||\varepsilon\, \nabla\left(\sqrt{\rho_{in}}\right) + i\, \sqrt{\rho_{in}}\, \nabla\phi_{in}||_{L^2}^2 + \int_{\mathbb{R}^d} \rho_{in}(x) \ln \rho_{in}(x)\, dx,$$

and translate this property into estimates on v_ε

$$\mathcal{E}_{\text{kin}} + \frac{\dot\tau^2}{2} \int |y|^2\, |v_\varepsilon|^2 - \varepsilon \frac{\dot\tau}{\tau} \operatorname{Im} \int v_\varepsilon(t,y)\, y\, \overline{\nabla v_\varepsilon}(t,y)\, dy + \lambda \int |v_\varepsilon|^2 \ln |v_\varepsilon|^2 - \lambda d\, ||\gamma^2||_{L^1} \ln \tau$$
$$+ 2\lambda\, ||\gamma^2||_{L^1} \ln\left(\frac{||\sqrt{\rho_{in}}||_{L^2}}{||\gamma||_{L^2}} \right) = \frac{1}{2}\, ||\varepsilon\, \nabla\left(\sqrt{\rho_{in}}\right) + i\, \sqrt{\rho_{in}}\, \nabla\phi_{in}||_{L^2}^2 + \int_{\mathbb{R}^d} \rho_{in}(x) \ln \rho_{in}(x)\, dx.$$

Therefore, we obtain

$$\left| \frac{\dot\tau^2}{2} \int |y|^2\, |v_\varepsilon|^2 - \lambda d\, ||\gamma^2||_{L^1} \ln\tau \right| \leq \left| \varepsilon \frac{\dot\tau}{\tau} \operatorname{Im} \int v_\varepsilon(t,y)\, y\, \overline{\nabla v_\varepsilon}(t,y)\, dy \right|$$
$$+ \left| \frac{1}{2}\, ||\varepsilon\, \nabla\left(\sqrt{\rho_{in}}\right) + i\, \sqrt{\rho_{in}}\, \nabla\phi_{in}||_{L^2}^2 + \int_{\mathbb{R}^d} \rho_{in}(x) \ln \rho_{in}(x)\, dx \right.$$
$$\left. -\lambda \int |v_\varepsilon|^2 \ln |v_\varepsilon|^2 - 2\lambda\, ||\gamma^2||_{L^1} \ln\left(\frac{||\sqrt{\rho_{in}}||_{L^2}}{||\gamma||_{L^2}} \right) \right|.$$

In the right hand side, the second group of terms is bounded thanks to the bounds previously found in (1.13) for the last two terms and the assumptions (1.11) (along with the considerations previously discussed) for the first two terms of this group. For the bound of the first term, we use $||v_\varepsilon(t)||_{L^2} = ||\gamma||_{L^2}$ and the fact that $\frac{\varepsilon}{\tau}\nabla v_\varepsilon$ is uniformly bounded in $L_t^\infty L_y^2$. It follows:

$$\left|\frac{\dot\tau^2(t)}{2}\int |y|^2\,|v_\varepsilon|^2 - \lambda d\,||\gamma^2||_{L^1}\ln\tau(t)\right| \le C(\dot\tau(t)+1),$$

with a constant $C > 0$ which does not depend on time and on ε. Moreover, the equation satisfied by τ leads to

$$\frac{\dot\tau^2}{2} = 2\lambda\ln\tau,$$

which gives in the above inequality for all $t > 1$

$$\left|\int |y|^2\,|v_\varepsilon|^2 - \frac{d}{2}||\gamma^2||_{L^1}\right| \le C\frac{\dot\tau(t)+1}{\dot\tau^2(t)}.$$

The last statement follows from the first two ones, and the fact that the L^2-norm is constant. □

Equation on ρ_ε

First, we define

$$\rho_\varepsilon = |v_\varepsilon|^2,$$
$$J_\varepsilon = \mathrm{Im}(\varepsilon\,\overline{v_\varepsilon}\,\nabla v_\varepsilon).$$

Then we compute

$$\partial_t\rho_\varepsilon + \frac{1}{\tau^2(t)}\nabla\cdot J_\varepsilon = 0 \qquad \text{in } \mathcal{D}'. \tag{2.2}$$

Moreover, in the same way as in [5], we also obtain

$$\partial_t J_\varepsilon + \lambda\,\nabla\rho_\varepsilon + 2\lambda\,y\,\rho_\varepsilon = \frac{\varepsilon^2}{4\,\tau^2(t)}\Delta\nabla\rho_\varepsilon - \frac{\varepsilon^2}{\tau^2(t)}\nabla\cdot(\mathrm{Re}(\nabla v_\varepsilon\otimes\overline{\nabla v_\varepsilon})). \tag{2.3}$$

Putting (2.2) and (2.3) together leads to

$$\partial_t(\tau^2\,\partial_t\rho_\varepsilon) = \lambda\,L\rho_\varepsilon - \frac{\varepsilon^2}{4\,\tau^2(t)}\Delta^2\rho_\varepsilon + \frac{\varepsilon^2}{\tau^2(t)}\nabla\cdot\left(\nabla\cdot(\mathrm{Re}(\nabla v_\varepsilon\otimes\overline{\nabla v_\varepsilon}))\right). \tag{2.4}$$

where $L = \Delta + \nabla\cdot(2y\,.)$ is a Fokker-Plank operator.

Change of time variable

Using the same change of time variable as in [5]

$$s = \frac{1}{2}\ln\dot\tau(t),$$

we obtain the same kind of equation (using the notation $f(t) = \check f(s(t))$ for the change of time variable)

$$\partial_s\check\rho_\varepsilon - \frac{2\lambda}{\dot{\check\tau}^2}\partial_s\check\rho_\varepsilon + \frac{\lambda}{\dot{\check\tau}^2}\partial_s^2\check\rho_\varepsilon = L\check\rho_\varepsilon - \frac{\varepsilon^2}{4\,\lambda\,\check\tau^2(s)}\Delta^2\check\rho_\varepsilon + \frac{\varepsilon^2}{\lambda\,\check\tau^2(s)}\nabla\cdot\left(\nabla\cdot(\mathrm{Re}(\nabla\check v_\varepsilon\otimes\overline{\nabla\check v_\varepsilon}))\right), \tag{2.5}$$

and the inequality (1.14) becomes

$$\int_0^\infty\left(\frac{\varepsilon\dot{\check\tau}(s)}{\check\tau(s)}\right)^2||\nabla\check v_\varepsilon(s)||_{L^2}^2\,ds \le C < \infty. \tag{2.6}$$

Limit as $s \to \infty$

Again, we will use the same strategy as in [5]. Take $s_n \underset{n\to\infty}{\to} \infty$. By de la Vallée-Poussin and Dunford-Pettis theorems with the bounds on $\check{\rho}_\varepsilon$, we have a subsequence $s_{\phi(n)}$ such that, using the notation f_n for $f_n(s) = f(s + s_{\phi(n)})$,

$$\check{\rho}_{\varepsilon,n} = \check{\rho}_\varepsilon(s + s_{\phi(n)}, y) \rightharpoonup \check{\rho}_\infty^\varepsilon \in L^\infty([-1,\infty), L^1_2 \cap L \log L) \qquad \text{in } L^p_{loc}([-1,\infty), L^1_y)$$

for every $p \in [1,\infty)$. Property (2.6) implies that

$$\frac{\varepsilon^2}{\check{\tau}_n^2(t)} \nabla \cdot \left(\nabla \cdot (\mathrm{Re}(\nabla \check{v}_{\varepsilon,n} \otimes \overline{\nabla \check{v}_{\varepsilon,n}}))\right) \underset{n\to\infty}{\longrightarrow} 0, \qquad \text{in } L^1_s(-1,\infty; W^{-2,1}(\mathbb{R}^d)).$$

Moreover, the L^1-bound on $\check{\rho}_\varepsilon(s)$ leads to

$$\frac{\varepsilon^2}{4\lambda\, \check{\tau}_n^2(t)} \Delta^2 \check{\rho}_{\varepsilon,n} \underset{n\to\infty}{\longrightarrow} 0, \qquad \text{in } L^\infty_s(-1,\infty; W^{-4,1}(\mathbb{R}^d)).$$

Passing the equation (2.5) to the weak limit yields

$$\partial_s \check{\rho}_\infty^\varepsilon = L \check{\rho}_\infty^\varepsilon.$$

Thanks to the bounds on $\check{\rho}_\infty$, we know that (see for instance [2, Corollary 2.17]) the solution actually converges to the usual Gaussian:

$$\lim_{s\to\infty} ||\check{\rho}_\infty^\varepsilon - \gamma^2||_{L^1(\mathbb{R}^d)} = 0.$$

The next step is to show that $\check{\rho}_\infty$ is actually independent of time. Come back to equation (2.2). We still have some bounds on J_ε thanks to (1.13) and (1.14), which are, for every $t \geq 0$ and $\varepsilon > 0$:

$$\frac{1}{\tau(t)} \int_{\mathbb{R}^d} (1 + |y|)|J_\varepsilon(t,y)|\, dy \leq C,$$
$$\int_0^\infty \frac{\dot{\tau}(t)}{\tau(t)} \left(\frac{1}{\tau(t)} ||J_\varepsilon(t)||_{L^1}\right)^2 dt \leq C.$$

In terms of s, those inequalities and the equation (2.2) read

$$\frac{1}{\check{\tau}(s)} \int_{\mathbb{R}^d} (1 + |y|)\, |\check{J}_\varepsilon(s,y)|\, dy \leq C,$$
$$\int_0^\infty \left(\frac{\dot{\check{\tau}}(s)}{\check{\tau}(s)} \int_{\mathbb{R}^d} |\check{J}_\varepsilon(s,y)| dy\right)^2 ds \leq C,$$
$$\partial_s \check{\rho}_\varepsilon + \frac{\dot{\check{\tau}}(s)}{\lambda\, \check{\tau}(s)} \nabla \cdot \check{J}_\varepsilon = 0. \tag{2.7}$$

Then, the second inequality implies

$$\frac{\dot{\check{\tau}}}{\check{\tau}} \check{J}_\varepsilon \in L^2_s L^1_y,$$

which yields

$$\frac{\dot{\check{\tau}}_n}{\check{\tau}_n} \nabla \cdot \check{J}_{\varepsilon,n} \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{in } L^2_{s,\mathrm{loc}}(-1,\infty; W^{-1,1}),$$

hence

$$\partial_s \check{\rho}_\infty^\varepsilon = 0, \tag{2.8}$$

which was needed. So $\check{\rho}_\infty^\varepsilon(s) = \gamma^2$ for all $s \in (-1,\infty)$. The limit being unique, we do not need any extraction:

$$\check{\rho}_\varepsilon(. + s) \underset{s\to\infty}{\rightharpoonup} \gamma^2 \qquad \text{in } L^p_{loc}([-1,\infty), L^1_y) \tag{2.9}$$

for every $p \in [1,\infty)$.

Punctual limit

The previous result shows that the weak limit γ^2 of $\check{\rho}_\varepsilon(.+s)$ in $L^p_{\text{loc}}([-1,\infty), L^1_y)$ for every $p \in [1,\infty)$ does not depend on time. Therefore, we can hope a weak punctual convergence, i.e. that $\check{\rho}_\varepsilon(s)$ converges weakly in L^1_y as $s \to \infty$ to γ^2. Such a result may only be true if the oscillation in time of $\check{\rho}_\varepsilon$ are not too strong, but this should actually be the case thanks to (2.7) since $\frac{\dot{\check{\tau}}}{\check{\tau}} \check{J}_\varepsilon$ has some integrability property: $\frac{\dot{\check{\tau}}}{\check{\tau}} \check{J}_\varepsilon \in L^2_s L^1_y$. Furthermore, the change of time variable has the following properties:

$$s(t) \underset{t\to\infty}{\sim} \frac{1}{4} \ln \ln t \underset{t\to\infty}{\longrightarrow} \infty.$$

Therefore, if we prove such a weak convergence for the time variable s, it will also prove the punctual convergence for the time variable t.

First of all, come back to the equation (2.7). Take $b \le a \le 0$, $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^d)$ and $\theta_n \in W^{1,\infty}(\mathbb{R})$ satisfying

$$0 \le \theta_n \le 1, \qquad \text{supp}\, \theta_n \subset [0, 2b] \qquad \theta_n(s) \underset{n\to\infty}{\longrightarrow} \mathbb{1}_{[a,b]}(s) \qquad \text{for all } s \in \mathbb{R}^+ \setminus \{a,b\},$$
$$\theta_n' \underset{n\to\infty}{\longrightarrow} \delta_a - \delta_b \qquad \text{in } \mathcal{M}(\mathbb{R}^+).$$

The equation (2.2) reads:

$$\iint_{\mathbb{R}^+_s \times \mathbb{R}^d} \check{\rho}_\varepsilon(s,y)\, \phi(y)\, \theta_n'(s)\, dyds = -\frac{1}{\lambda} \iint_{\mathbb{R}^+_s \times \mathbb{R}^d} \frac{\dot{\check{\tau}}(s)}{\check{\tau}(s)} \check{J}_\varepsilon(s,y)\, \nabla\phi(y)\, \theta_n(s)\, dyds.$$

The right-hand side converges as $n \to \infty$ thanks to dominated convergence:

$$\frac{\dot{\check{\tau}}(s)}{\check{\tau}(s)} \check{J}_\varepsilon(s,y)\, \nabla\phi(y)\, \theta_n(s) \underset{n\to\infty}{\longrightarrow} \check{J}_\varepsilon(s,y)\, \nabla\phi(y)\, \mathbb{1}_{[a,b]}(s) \qquad \text{for all } s \in \mathbb{R}^+ \setminus \{a,b\}, y \in \mathbb{R}^d,$$
$$\left| \frac{\dot{\check{\tau}}(s)}{\check{\tau}(s)} \check{J}_\varepsilon(s,y)\, \nabla\phi(y)\, \theta_n(s) \right| \le ||\nabla\phi||_{L^\infty} \frac{\dot{\check{\tau}}(s)}{\check{\tau}(s)} |\check{J}_\varepsilon|(s,y)\, \mathbb{1}_{[0,2b]}(s) \qquad \text{for all } s \in \mathbb{R}^+ \setminus \{a,b\}, y \in \mathbb{R}^d.$$

with $\frac{\dot{\check{\tau}}(s)}{\check{\tau}(s)} |\check{J}_\varepsilon|(s,y)\, \mathbb{1}_{[0,2b]}(s) \in L^1(\mathbb{R}^+_s \times \mathbb{R}^d)$. Therefore,

$$\iint_{\mathbb{R}^+_s \times \mathbb{R}^d} \frac{\dot{\check{\tau}}(s)}{\check{\tau}(s)} \check{J}_\varepsilon(s,y)\, \nabla\phi(y)\, \theta_n(s)\, dyds \underset{n\to\infty}{\longrightarrow} \int_a^b \int_{\mathbb{R}^d} \frac{\dot{\check{\tau}}(s)}{\check{\tau}(s)} \check{J}_\varepsilon(s,y)\, \nabla\phi(y)\, dyds.$$

For the left-hand side, we get

$$\iint_{\mathbb{R}^+_s \times \mathbb{R}^d} \check{\rho}_\varepsilon(s,y)\, \phi(y)\, \theta_n'(s)\, dyds = \int_0^\infty \left(\int_{\mathbb{R}^d} \check{\rho}_\varepsilon(s,y)\, \phi(y)\, dy \right) \theta_n'(s)\, ds,$$

and since $\check{\rho}_\varepsilon \in \mathcal{C}(\mathbb{R}^+_s, L^1(\mathbb{R}^d))$, we know that

$$\int_{\mathbb{R}^d} \check{\rho}_\varepsilon(s,y)\, \phi(y)\, dy \text{ is continuous in } s. \tag{2.10}$$

Therefore, we also get convergence for this term thanks to the convergence of θ_n':

$$\iint_{\mathbb{R}^+_s \times \mathbb{R}^d} \check{\rho}_\varepsilon(s,y)\, \phi(y)\, \theta_n'(s)\, dyds \underset{n\to\infty}{\longrightarrow} \int_0^\infty \left(\int_{\mathbb{R}^d} \check{\rho}_\varepsilon(s,y)\, \phi(y)\, dy \right) d(\delta_a - \delta_b)$$
$$= \int_{\mathbb{R}^d} \check{\rho}_\varepsilon(a,y)\, \phi(y)\, dy - \int_{\mathbb{R}^d} \check{\rho}_\varepsilon(b,y)\, \phi(y)\, dy.$$

Thus, we get:

$$\int_{\mathbb{R}^d} \check{\rho}_\varepsilon(b,y)\,\phi(y)\,dy - \int_{\mathbb{R}^d} \check{\rho}_\varepsilon(a,y)\,\phi(y)\,dy = \frac{1}{\lambda}\int_a^b \int_{\mathbb{R}^d} \frac{\dot{\check{\tau}}(s)}{\check{\tau}(s)} \check{J}_\varepsilon(s,y)\,\nabla\phi(y)\,dyds. \tag{2.11}$$

Now, in the same way as previously, we have a sequence $(s_n)_n$ such that $s_n \underset{n\to\infty}{\longrightarrow} \infty$ and $\check{\rho}^\varepsilon_{\infty,0} \in L^1(\mathbb{R}^d)$ such that $\check{\rho}_\varepsilon(s_n) \underset{n\to\infty}{\rightharpoonup} \check{\rho}^\varepsilon_{\infty,0}$ in $L^1(\mathbb{R}^d)$. Take again $\phi \in W^{1,\infty}(\mathbb{R}^d)$, and $\psi \in \mathcal{C}^\infty_c((0,1))$ such that $\psi \geq 0$ and $\int_0^1 \psi(s)\,ds = 1$. Then

$$\int_{\mathbb{R}^d} \check{\rho}_\varepsilon(s_n,y)\,\phi(y)\,dy = \int_{\mathbb{R}^d} \check{\rho}_\varepsilon(s_n,y)\,\int_0^1 \phi(y)\,\psi(s)\,ds\,dy.$$

Therefore

$$\begin{aligned}
\int_{\mathbb{R}^d} \check{\rho}_\varepsilon(s_n,y)\,\phi(y)\,dy - \int_0^1\int_{\mathbb{R}^d} \check{\rho}_\varepsilon(s+s_n,y)\,\phi(y)\,\psi(s)\,dyds& \\
&= \int_0^1 \int_{\mathbb{R}^d} (\check{\rho}_\varepsilon(s_n,y) - \check{\rho}_\varepsilon(s_n+s,y))\,\phi(y)\,\psi(s)\,dyds \\
&= \int_0^1 \left(\int_{\mathbb{R}^d} (\check{\rho}_\varepsilon(s_n,y) - \check{\rho}_\varepsilon(s_n+s,y))\,\phi(y)\,dy\right)\psi(s)\,ds.
\end{aligned}$$

But then, thanks to (2.11), for every $s \in (0,1)$,

$$\int_{\mathbb{R}^d} (\check{\rho}_\varepsilon(s_n,y) - \check{\rho}_\varepsilon(s_n+s,y))\,\phi(y)\,dy = -\frac{1}{\lambda}\int_{s_n}^{s_n+s}\int_{\mathbb{R}^d} \frac{\dot{\check{\tau}}(r)}{\check{\tau}(r)}\check{J}_\varepsilon(r,y)\,\nabla\phi(y)\,dydr,$$

which yields

$$\left|\int_{\mathbb{R}^d} (\check{\rho}_\varepsilon(s_n,y) - \check{\rho}_\varepsilon(s_n+s,y))\,\phi(y)\,dy\right| \leq \frac{1}{\lambda}\int_{s_n}^{s_n+s}\int_{\mathbb{R}^d} \frac{\dot{\check{\tau}}(r)}{\check{\tau}(r)}|\check{J}_\varepsilon|(r,y)\,||\nabla\phi||_{L^\infty}\,dydr.$$

Therefore, we get

$$\begin{aligned}
\left|\int_{\mathbb{R}^d} \check{\rho}_\varepsilon(s_n,y)\,\phi(y)\,dy - \int_0^1\int_{\mathbb{R}^d} \check{\rho}_\varepsilon(s+s_n,y)\,\phi(y)\,\psi(s)\,dyds\right|& \\
&\leq \frac{1}{\lambda}\int_0^1\int_0^s\int_{\mathbb{R}^d} \frac{\dot{\check{\tau}}(s_n+r)}{\check{\tau}(s_n+r)}|\check{J}_\varepsilon|(s_n+r,y)\,||\nabla\phi||_{L^\infty}\,dydrds \\
&\leq \frac{||\nabla\phi||_{L^\infty}}{\lambda}\int_0^1\int_{\mathbb{R}^d} \frac{\dot{\check{\tau}}(s_n+r)}{\check{\tau}(s_n+r)}|\check{J}_\varepsilon|(s_n+r,y)\,dydr \\
&\leq \frac{||\nabla\phi||_{L^\infty}}{\lambda}\left(\int_0^1\left(\frac{\dot{\check{\tau}}(s_n+r)}{\lambda\,\check{\tau}(s_n+r)}\int_{\mathbb{R}^d}|\check{J}_\varepsilon|(s_n+r,y)\,dy\right)^2 dr\right)^{\frac{1}{2}} \\
&\leq ||\nabla\phi||_{L^\infty}\left(\int_{s_n}^{s_n+1}\left(\frac{\dot{\check{\tau}}(s)}{\lambda\,\check{\tau}(s)}\int_{\mathbb{R}^d}|\check{J}_\varepsilon|(s,y)\,dy\right)^2 ds\right)^{\frac{1}{2}} \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

since $\frac{\dot{\check{\tau}}(s)}{\check{\tau}(s)}\check{J} \in L^2_s L^1_y$. But then, thanks to the weak convergence of $\check{\rho}_\varepsilon$, we know that

$$\int_{\mathbb{R}^d} \check{\rho}_\varepsilon(s_n,y)\,\phi(y)\,dy \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \check{\rho}^\varepsilon_{\infty,0}(y)\,\phi(y)\,dy,$$

and, since $\phi(y)\,\psi(s) \in L^2(-1,\infty; L^\infty(\mathbb{R}^d))$ with support in time in $(0,1)$,

$$\int_0^1 \int_{\mathbb{R}^d} \tilde{\rho}_\varepsilon(s+s_n, y)\,\phi(y)\,\psi(s)\,dyds \underset{n\to\infty}{\longrightarrow} \int_0^1 \int_{\mathbb{R}^d} \gamma^2(y)\,\phi(y)\,\psi(s)\,dyds = \int_{\mathbb{R}^d} \gamma^2(y)\,\phi(y)\,dy.$$

Thus, for all $\phi \in W^{1,\infty}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \check{\rho}^\varepsilon_{\infty,0}(y)\,\phi(y)\,dy = \int_{\mathbb{R}^d} \gamma^2(y)\,\phi(y)\,dy.$$

This yields $\check{\rho}^\varepsilon_{\infty,0} = \gamma^2$ and hence, the limit being unique,

$$\check{\rho}_\varepsilon(s) \underset{s\to\infty}{\rightharpoonup} \gamma^2, \qquad \text{in } L^1(\mathbb{R}^d).$$

This limit can then be rewritten in terms of t:

$$\rho_\varepsilon(t) \underset{t\to\infty}{\rightharpoonup} \gamma^2, \qquad \text{in } L^1(\mathbb{R}^d).$$

This ends the proof of Theorem 1.4. □

Remark 2.3. We see that those uniform estimations and convergence for the quadratic quantities still hold for more general initial data $u_{\varepsilon,in}$. The assumption we need is actually

$$\varepsilon\,||\nabla u_{\varepsilon,in}||_{L^2} + ||u_{\varepsilon,in}||_{L^2} + \int |u_{\varepsilon,in}|^2 \left|\ln |u_{\varepsilon,in}|^2\right| + ||u_{\varepsilon,in}|y|||_{L^2} \leq C, \tag{2.12}$$

for some constant C independent of ε. The convergence to the Gaussian is true for any $\varepsilon > 0$ and any $u_{\varepsilon,in}$, but to have some kind of "uniformity" in such a convergence, we should also need the assumption (2.12).

2.2 Wigner Transform and Wigner Measure

2.2.1 Definition and general properties

The Wigner Transform and the Wigner Measure are very useful tools allowing to make the mathematical link between Quantum Mechanics and Classic Mechanics in physics, more known as the semi-classical limit. They have already been studied a lot, initially by P.L. Lions and T. Paul. We will recall its definition and some properties which are significant for us, and we will refer to [12, 3, 11, 10] for more precision.

First, we recall its definition already given in (1.7).

Definition 2.1. *The Wigner Transform of a sequence $u_\varepsilon \in L^2(\mathbb{R}^d)$ for $\varepsilon > 0$ is defined by*

$$W_\varepsilon(x,\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^N} e^{-i\,\xi\cdot z}\, u_\varepsilon\left(x + \frac{\varepsilon z}{2}\right) \overline{u_\varepsilon\left(x - \frac{\varepsilon z}{2}\right)}\, dz = \mathcal{F}_z \tilde{\rho}_\varepsilon(x,\xi),$$

where

$$\tilde{\rho}_\varepsilon(x,z) = u_\varepsilon\left(x + \frac{\varepsilon z}{2}\right) \overline{u_\varepsilon\left(x - \frac{\varepsilon z}{2}\right)}. \tag{2.13}$$

If $u_\varepsilon \in L^2(\mathbb{R}^d)$, then for every $x \in \mathbb{R}^d$, $\tilde{\rho}_\varepsilon$ is in L^1. Therefore the Wigner Transform W_ε is well-defined. A first interesting property easily comes:

Proposition 2.1. *The Wigner Transform W_ε is real-valued.*

This shows that the Wigner Transform may act in some sense like a density, if we get some non-negativity. Such non-negativity may come from the fact that $\tilde{\rho}_\varepsilon(x,0) = |u_\varepsilon(x)|^2 \geq 0$, a property that must be taken into account when $\varepsilon \to 0$ when we look at the definition of the Wigner Transform. Before trying to translate such formal properties, we get some integrability and regularity results for the Wigner Transform.

Lemma 2.3. *For $u_\varepsilon \in L^2(\mathbb{R}^d)$, for every $\varepsilon > 0$, recall the definition of $\tilde{\rho}_\varepsilon$ given in* (2.13)*. It verifies*

$$\tilde{\rho}_\varepsilon \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \cap \mathcal{C}_0(\mathbb{R}^d_x, L^1(\mathbb{R}^d_z)) \cap \mathcal{C}_0(\mathbb{R}^d_z, L^1(\mathbb{R}^d_x)).$$

Therefore, the inverse Fourier Transform in x of the Wigner Transform W_ε is well-defined and is given by

$$(\mathcal{F}_x^{-1} W_\varepsilon)(\zeta,\xi) = \left(\frac{2\pi}{\varepsilon}\right)^d \hat{u_\varepsilon}\left(\frac{\xi}{\varepsilon} - \frac{\zeta}{2}\right) \overline{\hat{u_\varepsilon}\left(\frac{\xi}{\varepsilon} + \frac{\zeta}{2}\right)},$$

where $\hat{u_\varepsilon} = \mathcal{F}u_\varepsilon \in L^2(\mathbb{R}^d)$ is the Fourier Transform of u_ε.

Thus the Wigner Transform W_ε of u_ε verifies

$$W_\varepsilon \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \cap \mathcal{C}_0(\mathbb{R}^d_x, \mathcal{F}L^1(\mathbb{R}^d_\xi)) \cap \mathcal{C}_0(\mathbb{R}^d_\xi, \mathcal{F}L^1(\mathbb{R}^d_x)).$$

Formally, for $u_\varepsilon \in L^2(\mathbb{R}^d)$ and W_ε its Wigner Transform, one may have

$$\int_{\mathbb{R}^d} W_\varepsilon(x,\xi)\, d\xi = \int_{\mathbb{R}^d} \mathcal{F}_z \tilde{\rho}_\varepsilon(x,\xi)\, d\xi = \mathcal{F}_\xi^{-1}\left(\mathcal{F}_z \tilde{\rho}_\varepsilon(x,\xi)\right)(0) = \tilde{\rho}_\varepsilon(x,0) = |u_\varepsilon(x)|^2, \tag{2.14}$$

and in the same way, for example, if $u_\varepsilon \in H^2$,

$$\int_{\mathbb{R}^d} |\xi|^2\, W_\varepsilon(x,\xi)\, d\xi = \frac{\varepsilon^2}{4}\left(2\,|\nabla u_\varepsilon(x)|^2 - \Delta u_\varepsilon(x)\, \overline{u_\varepsilon(x)} - u_\varepsilon(x)\, \overline{\Delta u_\varepsilon(x)}\right), \tag{2.15}$$

and thus

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} |\xi|^2\, W_\varepsilon(x,\xi)\, d\xi dx = \varepsilon^2 \int_{\mathbb{R}^d} |\nabla u_\varepsilon(x)|^2\, dx, \tag{2.16}$$

which may also be true if $u_\varepsilon \in H^1$. However, we don't have any result of integrability in ξ for W_ε in general. A convolution with some good kernel may fix this problem, in order to get such properties. The first idea is to get a Gaussian kernel. In order to get the same convergence when $\varepsilon \to 0$ as W_ε, we would like such kernels to be mollifiers and approximate identity when $\varepsilon \to 0$. For this purpose, we define the Gaussian with ε variance by

$$\gamma_\varepsilon(x) = \frac{1}{(\pi\varepsilon)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{\varepsilon}\right),$$
$$G_\varepsilon(x,\xi) = \gamma_\varepsilon(x)\, \gamma_\varepsilon(\xi).$$

This leads to the definition of the Husimi Transform.

Definition 2.2. *The Husimi Transform of a sequence of functions $u_\varepsilon \in L^2(\mathbb{R}^d)$ for $\varepsilon > 0$ is defined by*

$$W_\varepsilon^H = W_\varepsilon * G_\varepsilon = W_\varepsilon *_x \gamma_\varepsilon *_\xi \gamma_\varepsilon.$$

This tool fixes the problem of integrability, but it also takes into account the question of non-negativity previously asked.

Lemma 2.4. *For every $u_\varepsilon \in L^2(\mathbb{R}^d)$, $\varepsilon > 0$, its Husimi Transform W_ε^H is non-negative and has the following properties of integration and regularity:*

$$W_\varepsilon^H \in W^{\infty,1}(\mathbb{R}^d \times \mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}_x^d, W^{\infty,1}(\mathbb{R}_\xi^d)) \cap \mathcal{C}^\infty(\mathbb{R}_\xi^d, W^{\infty,1}(\mathbb{R}_x^d)).$$

Thanks to the non-negativity of the Husimi Transform and its high regularity, we can then face the problem of the formal estimates (2.14)-(2.16) in terms of the Husimi Transform.

Lemma 2.5. *For every $u_\varepsilon \in L^2(\mathbb{R}^d)$, $\varepsilon > 0$, and its Husimi Transform W_ε^H, there holds:*

$$\mathcal{F}_\xi W_\varepsilon^H(x,z) = \frac{1}{(2\pi)^d}\,(\tilde{\rho}_\varepsilon(.,-z) * \gamma_\varepsilon)(x)\,\exp\left(-\frac{\varepsilon\,|z|^2}{4}\right).$$

This equality is very useful to actually compute some integrals in ξ involving the Husimi Transform.

Proposition 2.2. *Take $u_\varepsilon \in L^2(\mathbb{R}^d)$, $\varepsilon > 0$ and denote its Husimi Transform W_ε^H. For every $x \in \mathbb{R}^d$, there holds:*

1.
$$\int_{\mathbb{R}^d} W_\varepsilon^H(x,\xi)\,d\xi = |u_\varepsilon|^2 * \gamma_\varepsilon(x), \tag{2.17}$$
and
$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} W_\varepsilon^H(x,\xi)\,d\xi dx = ||u_\varepsilon||_{L^2}^2. \tag{2.18}$$

2. *if $u_\varepsilon \in H^1(\mathbb{R}^d)$,*
$$\int_{\mathbb{R}^d} |\xi|^2\,W_\varepsilon^H(x,\xi)\,d\xi = \varepsilon^2\,|\nabla u_\varepsilon|^2 * \gamma_\varepsilon(x) - \frac{\varepsilon^2}{4}\,|u_\varepsilon|^2 * \Delta\gamma_\varepsilon(x) + \frac{\varepsilon d}{2}|u_\varepsilon|^2 * \gamma_\varepsilon(x), \tag{2.19}$$
and
$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} |\xi|^2\,W_\varepsilon^H(x,\xi)\,d\xi dx = \varepsilon^2||\nabla u_\varepsilon||_{L^2}^2 + \frac{\varepsilon d}{2}||u_\varepsilon||_{L^2}^2. \tag{2.20}$$
In a more general way,
$$\int_{\mathbb{R}^d} \xi_i\xi_j\,W_\varepsilon^H(x,\xi)\,d\xi = \varepsilon^2\,\operatorname{Re}\left(\partial_i u_\varepsilon\,\overline{\partial_j u_\varepsilon}\right) * \gamma_\varepsilon(x) - \frac{\varepsilon^2}{4}\,|u_\varepsilon|^2 * \partial_i\partial_j\gamma_\varepsilon(x) + \frac{\varepsilon\delta_{ij}}{2}|u_\varepsilon|^2 * \gamma_\varepsilon(x),$$
and
$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \xi_i\xi_j\,W_\varepsilon^H(x,\xi)\,d\xi dx = \varepsilon^2\int_{\mathbb{R}^d} \operatorname{Re}\left(\partial_i u_\varepsilon\,\overline{\partial_j u_\varepsilon}\right)\,dx + \frac{\varepsilon\delta_{ij}}{2}||u_\varepsilon||_{L^2}^2.$$

3. *if $u_\varepsilon \in H^1(\mathbb{R}^d)$,*
$$\int_{\mathbb{R}^d} \xi\,W_\varepsilon^H(x,\xi)\,d\xi = \varepsilon\,\operatorname{Im}(\nabla u_\varepsilon\,\overline{u_\varepsilon}) * \gamma_\varepsilon\,(x)\,, \tag{2.21}$$
and therefore
$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \xi\,W_\varepsilon^H(x,\xi)\,d\xi dx = \int_{\mathbb{R}^d} \varepsilon\,\operatorname{Im}(\nabla u_\varepsilon\,\overline{u_\varepsilon})\,dx. \tag{2.22}$$

4. *if $u_\varepsilon \in \mathcal{F}(H^1)$,*
$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x|^2\,W_\varepsilon^H(x,\xi)\,dxd\xi = ||x\,u_\varepsilon||_{L^2}^2 + \frac{\varepsilon d}{2}\,||u_\varepsilon||_{L^2}^2. \tag{2.23}$$

The first part of this proposition can be found in [12], whereas the other parts (which are based on the same idea with some extra calculus) will be done in Appendix A.

Thanks to those estimations, we see that if $(u_\varepsilon)_\varepsilon$ is bounded in $L^2(\mathbb{R}^d)$, then $(W_\varepsilon^H)_\varepsilon$ is a sequence of non-negative functions bounded in $L^1(\mathbb{R}^d)$. On the other hand, bounds on $(W_\varepsilon)_\varepsilon$ are less obvious. A way to get some consists in defining this space of test functions.

$$\mathcal{A} = \left\{ \phi \in \mathcal{C}_0(\mathbb{R}^d_x \times \mathbb{R}^d_\xi), (\mathcal{F}_\xi \phi)(x,z) \in L^1(\mathbb{R}^d_z, \mathcal{C}_0(\mathbb{R}^d_x)) \right\}$$

endowed with the norm

$$||\phi||_{\mathcal{A}} = ||\mathcal{F}_\xi \phi||_{L^1_z L^\infty_x}$$

which makes it a Banach space and algebra.

Proposition 2.3. *If $(u_\varepsilon)_\varepsilon$ is bounded in $L^2(\mathbb{R}^d)$ then W_ε is bounded in $\mathcal{A}'$.*

Then, if $(u_\varepsilon)_\varepsilon$ is bounded in $L^2(\mathbb{R}^d)$, up to extracting a subsequence, we can suppose that $W_\varepsilon \rightharpoonup W$ in $\mathcal{A}'$ (endowed with the weak-$*$ topology) but also $W_\varepsilon^H \rightharpoonup W^H$ in $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$ thanks to the L^1 bound on W_ε^H and the fact that those functions are non-negative.

Then, we would like such limits to have some similar properties as we calculated before, for example (2.17) and (2.18). In some way, we do not want to have loss of mass at infinity or due to oscillatory behaviour. This motivates the following definitions.

Definition 2.3. *A sequence $(u_\varepsilon)_\varepsilon$ in $L^2(\mathbb{R}^d)$ is said to be:*

- *ε-oscillatory if* $\sup_\varepsilon \int_{|\xi| \geq \frac{R}{\varepsilon}} |\hat{u}_\varepsilon(\xi)|^2 \, d\xi \underset{R \to +\infty}{\longrightarrow} 0.$
- *compact at infinity if* $\sup_\varepsilon \int_{|x| \geq R} |u_\varepsilon(x)|^2 \, dx \underset{R \to +\infty}{\longrightarrow} 0.$

Remark 2.4. Note that the condition

$$\exists s > 0 \text{ such that } \varepsilon^s D^s u_\varepsilon \text{ is uniformly bounded in } L^2_{loc}$$

(where $D^s f = \mathcal{F}^{-1}(|\xi|^s \mathcal{F} f)$) is sufficient for being ε-oscillatory.

In the same way, the condition

$$\exists g \in \left\{ f : \mathbb{R}^+ \to \mathbb{R}^+, f(t) \underset{t \to +\infty}{\longrightarrow} +\infty \right\} \text{ such that } g(|x|) u_\varepsilon \text{ is uniformly bounded in } L^2(\mathbb{R}^d)$$

is sufficient for being compact at infinity.

We can now state the main theorem that we will use. This theorem relates some formal properties on W_ε (like (2.17)), the Husimi transform (which should converge as $\varepsilon \to 0$ to the same limit as W_ε in view of its definition) and the final limit.

Theorem 2.2 ([12, Theorem III.1.]). *1.* $W = W^H \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$.

2. *If $|u_\varepsilon(x)|^2$ (resp. $\left|\hat{u_\varepsilon}\left(\frac{\xi}{\varepsilon}\right)\right|^2 (2\pi\varepsilon)^d$) weakly converges in measure to a non-negative measure μ_x (resp. μ_ξ), then $\mu_x(x) \geq \int_{\mathbb{R}^d} W(x, d\xi)$ (resp. $\mu_\xi(\xi) \geq \int_{\mathbb{R}^d} W(dx, \xi)$).*

   For μ_x, the inequality is an equality iff (u_ε) is ε-oscillatory.

3. *We have the equality*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} W(dx,d\xi) = \lim_{\varepsilon\to 0}\int |u_\varepsilon(x)|^2\,dx$$

if and only if (u_ε) *is* ε*-oscillatory and compact at infinity.*

Thanks to the first part of this theorem, we see that W is actually a (non-negative) measure, following the intuition we had previously. We call W a Wigner Measure.

Remark 2.5. There is in general no reason for W to be unique. This is why we call W a Wigner Measure and not *the* Wigner Measure. However, in some case, it can be proven that it is unique and in such cases we can call it the Wigner Measure of the sequence.

Remark 2.6. ε is only the parameter defining the sequence (u_ε), so its appearance in the definition of the Wigner Transform (and then in the Husimi Transform) is purely arbitrary. Actually, in order to have good results, which means non-trivial measure, ε should actually be of the same order as the characteristic oscillatory length, as the fact of being ε-oscillatory predicts it.

Example 2.1 (WKB State)**.** *If* $u_\varepsilon = f(x)\,e^{i\frac{\phi_{in}(x)}{\varepsilon}}$ *where* $f\in L^2(\mathbb{R}^d)$ *and* $\phi_{in}\in W^{1,1}_{loc}(\mathbb{R}^d)$*, then:*

- $\tilde{\rho}_\varepsilon$ *converges to* $|f(x)|^2\,e^{i\,\nabla\phi_{in}(x)\cdot z}$ *in* $\mathcal{S}'(\mathbb{R}^d\times\mathbb{R}^d)$.
- *the Wigner transform converges to* $W = |f(x)|^2 dx\otimes\delta_{\xi=\nabla\phi_{in}(x)}$.

Remark 2.7. This example motivates us in the sense that our intuition (and the properties that we got) leads us to think that the solution to (1.3) may be described very formally as $u_\varepsilon(t,x) = a_\varepsilon(t,x)\,e^{i\frac{\phi_\varepsilon(t,x)}{\varepsilon}}$, and some convergence for a_ε and ϕ_ε for $\varepsilon\to 0$ would yield a mono-kinetic Wigner measure.

In order to be able to apply a semi-classical limit, we should now introduce some dependence in time. For this purpose, we take $u_\varepsilon = u_\varepsilon(t,x)\in\mathcal{C}_b([0,T),L^2(\mathbb{R}^d))$ a sequence of function depending on time, and we define its Wigner transform $W_\varepsilon(t)$ by the Wigner transform of $u_\varepsilon(t)$ for every t.

Proposition 2.4. *Take* $u_\varepsilon = u_\varepsilon(t,x)\in\mathcal{C}_b([0,T),L^2(\mathbb{R}^d))$ *(*$T\in(0,\infty]$*) such that there exists a real-valued potential* $V_\varepsilon\in L^\infty([0,T),\mathcal{F}(L^p)(\mathbb{R}^d))$ *for some* $p\in[1,\infty)$ *such that*

$$i\varepsilon\partial_t u_\varepsilon + \frac{\varepsilon^2}{2}\Delta u_\varepsilon = V_\varepsilon u_\varepsilon \qquad \text{in } \mathcal{D}'.$$

Then W_ε *verifies*

$$\partial_t W_\varepsilon + \xi\cdot\nabla_x W_\varepsilon + K_\varepsilon *_\xi W_\varepsilon = 0,$$

where

$$K_\varepsilon(t,x,\xi) = \frac{i}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\xi\cdot y}\,\frac{V_\varepsilon\left(t,x+\frac{\varepsilon y}{2}\right) - V_\varepsilon\left(t,x-\frac{\varepsilon y}{2}\right)}{\varepsilon}\,dy.$$

Remark 2.8. The hypothesis $V_\varepsilon\in L^\infty([0,T),\mathcal{F}(L^p(\mathbb{R}^d)))$ is not the sharpest one, and it can be weakened (see for instance [12, Proposition II.1.]). The only issue that we have to face regarding this result is to have K_ε well-defined in some sense (for example, in $L^\infty([0,T),\mathcal{C}(\mathbb{R}^d_x,L^p(\mathbb{R}^d_\xi)))$, or in $L^\infty([0,T),\mathcal{C}(\mathbb{R}^d_x,H^{-p}(\mathbb{R}^d_\xi)))$ in [12]), with also the convolution $K_\varepsilon *_\xi W_\varepsilon$ which needs to be well-defined (for instance in $\mathcal{S}'$).

Remark 2.9. Note that V_ε may depend on u_ε itself without having any problem.

Formally, we see that K_ε should converge (in some sense) to

$$K_0 = i\,\nabla V_0(x) \cdot \mathcal{F}(y) = -\nabla V_0(x) \cdot \nabla \delta_0(\xi).$$

Many results are going in this direction, for example:

Proposition 2.5. *If $(u_\varepsilon = u_\varepsilon(t,x))_{\varepsilon>0}$ is a sequence bounded in $\mathcal{C}([0,T), L^2(\mathbb{R}^d))$ ($T \in (0,\infty]$) uniformly in ε and $V_0 \in \mathcal{C}([0,T), L^2(\mathbb{R}^d) \cap \mathcal{C}^1(\mathbb{R}^d))$ is a real-valued potential satisfying* (1.8)*:*

$$i\varepsilon\partial_t u_\varepsilon + \frac{\varepsilon^2}{2}\Delta u_\varepsilon = V_0 u_\varepsilon \qquad \text{in } \mathcal{D}',$$

then, up to a subsequence, the Wigner Transform W_ε of $(u_\varepsilon)_\varepsilon$ converges uniformly on every compact of $[0,T)$ in $\mathcal{A}'_{w-*}$ to $W \in \mathcal{C}_b([0,T), \mathcal{M}_{w-*})$ which verifies

$$\partial_t W + \xi \cdot \nabla_x W + \nabla_x V_0 \cdot \nabla_\xi W = 0.$$

Remark 2.10. Again, the hypothesis on V_0 can be weakened (see for instance [12, Théorème IV.1. and Remarque IV.2.]).

We see that the previous result is made in the linear case, but it also works on some non-linear case:

Proposition 2.6. *Take $V_0 \in \mathcal{C}^1(\mathbb{R}^d)$ such that $V_0 \geq C > -\infty$, $\nabla V_0 \in \mathcal{C}_b(\mathbb{R}^d)$. If $(u_\varepsilon = u_\varepsilon(t,x))_\varepsilon$ is a sequence uniformly bounded in ε in $\mathcal{C}([0,T), L^2(\mathbb{R}^d))$ which verifies*

$$i\varepsilon\partial_t u_\varepsilon + \frac{\varepsilon^2}{2}\Delta u_\varepsilon = V_\varepsilon u_\varepsilon \qquad \text{in } \mathcal{D}',$$
$$V_\varepsilon = V_0 * |u_\varepsilon|^2,$$

then, up to a subsequence, the Wigner Transform W_ε of $(u_\varepsilon)_\varepsilon$ converges uniformly on every compact of $[0,T)$ in $\mathcal{A}'_{w-*}$ to $W \in \mathcal{C}_b([0,T), \mathcal{M}_{w-*})$ which verifies

$$\partial_t W + \xi \cdot \nabla_x W + \nabla_x V \cdot \nabla_\xi W = 0, \tag{2.24}$$
$$V = V_0 * \rho, \qquad \rho(t,x) = \int_{\mathbb{R}^d} f(t,x,d\xi). \tag{2.25}$$

Remark 2.11. Such result works with some other hypothesis on V_0, for example [12, Théorème IV.2.].

Remark 2.12. We gave some results about the weak convergence of the Wigner Transform. Under some stronger hypothesis on the potential of the Schrödinger equation for the linear case, A. Athanassoulis and T. Paul have shown some strong convergence results ([3]). However, there are also some cases where we know that there is no convergence

Those results are rather positive for our point of view. They show that we could have a link between Logarithmic Schrödinger Equation and Logarithmic Vlasov Equation thanks to the Wigner Transform and its limit when $\varepsilon \to 0$. However, this link is still formal, and the proof to make it rigorous must be difficult because of the very little regularity of our potential $\ln |u_\varepsilon|^2$, which depends on the solution u_ε (more exactly on $|u_\varepsilon|^2$) and may have no meaning in L^1.

2.2.2 Wigner Measure of a Non-Linear Schrödinger Equation with logarithmic non-linearity

We are now interested in the limit of the Wigner Transform of $(u_\varepsilon)_{\varepsilon>0}$, where u_ε is the solution in $L^\infty(\mathbb{R}^+, \mathcal{F}(H^1)\cap H^1)\cap \mathcal{C}(\mathbb{R}^+, L^2\cap H^1_{\mathrm{w}})$ of (1.3) with $u_{\varepsilon,in} = \sqrt{\rho_{in}(x)}\, e^{i\,\frac{\phi_{in}(x)}{\varepsilon}}$:

$$i\,\varepsilon\,\partial_t u_\varepsilon + \frac{\varepsilon^2}{2}\Delta u_\varepsilon = \lambda u_\varepsilon \ln|u_\varepsilon|^2, \qquad u_\varepsilon(0,x) = u_{\varepsilon,in} = \sqrt{\rho_{in}(x)}\, e^{i\,\frac{\phi_{in}(x)}{\varepsilon}},$$

where $\lambda > 0$, and $\rho_{in}\geq 0$ and ϕ_{in} are given functions satisfying (1.11):

$$\sqrt{\rho_{in}} \in \mathcal{F}(H^1)\cap H^1(\mathbb{R}^d)\setminus\{0\}\,, \qquad \phi_{in}\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d), \qquad \sqrt{\rho_{in}}\,\nabla\phi_{in}\in L^2(\mathbb{R}^d),$$

so that $u_{\varepsilon,in}\in\mathcal{F}(H^1)\cap H^1$ for all $\varepsilon>0$. Recall the definition of v_ε made in (1.12)

$$u_\varepsilon(t,x) = \frac{1}{\tau(t)^{\frac{d}{2}}}\,\frac{||\sqrt{\rho_{in}}||_{L^2}}{||\gamma||_{L^2}}\, v_\varepsilon\left(t, \frac{x}{\tau(t)}\right)\, e^{i\,\frac{\dot{\tau}(t)}{\tau(t)}\frac{|x|^2}{2\,\varepsilon}},$$

In this section, we denote by W_ε (resp. $\tilde{W}_\varepsilon$) the Wigner Transform of u_ε (resp. v_ε). Foremost, we see that (1.18) and (1.19) are a straightforward consequence of Example 2.1.

Link between the two Wigner Transforms

We first prove the relation (1.20) between the two Wigner Transform.

$$\begin{aligned}
W_\varepsilon(t,x,\xi) &= \frac{1}{(2\pi)^d}\int e^{i\,y\cdot\xi} u_\varepsilon\left(t, x+\frac{\varepsilon}{2}\,y\right)\overline{u_\varepsilon\left(t,x-\frac{\varepsilon}{2}\,y\right)}\,dy\\
&= \frac{||\rho_{in}||_{L^1}}{||\gamma^2||_{L^1}}\frac{1}{(2\pi\,\tau(t))^d}\int e^{i\,y\cdot\xi} v_\varepsilon\left(t,\frac{x+\frac{\varepsilon}{2}\,y}{\tau(t)}\right)\overline{v_\varepsilon\left(t,\frac{x-\frac{\varepsilon}{2}\,y}{\tau(t)}\right)}\,e^{-i\,\frac{\dot{\tau}(t)}{\tau(t)}\,\frac{|x-\frac{\varepsilon}{2}\,y|^2-|x+\frac{\varepsilon}{2}\,y|^2}{2\varepsilon}}\,dy\\
&= \frac{||\rho_{in}||_{L^1}}{||\gamma^2||_{L^1}}\frac{1}{(2\pi\,\tau(t))^d}\int e^{i\,y\cdot\xi - i\frac{\dot{\tau}(t)}{\tau(t)}\,x\cdot y} v_\varepsilon\left(t,\frac{x+\frac{\varepsilon}{2}\,y}{\tau(t)}\right)\overline{v_\varepsilon\left(t,\frac{x-\frac{\varepsilon}{2}\,y}{\tau(t)}\right)}\,dy\\
&= \frac{||\rho_{in}||_{L^1}}{||\gamma^2||_{L^1}}\frac{1}{(2\pi)^d}\int e^{i\,\tau(t)\,z\cdot\xi - i\frac{\dot{\tau}(t)}{\tau(t)}\,x\cdot y} v_\varepsilon\left(t,\frac{x}{\tau(t)}+\frac{\varepsilon}{2}\,z\right)\overline{v_\varepsilon\left(t,\frac{x}{\tau(t)}-\frac{\varepsilon}{2}\,z\right)}\,dz\\
&= \frac{||\rho_{in}||_{L^1}}{||\gamma^2||_{L^1}}\,\tilde{W}_\varepsilon\left(t,\frac{x}{\tau(t)},\tau(t)\,\xi-\dot{\tau}(t)\,x\right).
\end{aligned}$$

This link between W_ε and $\tilde{W}_\varepsilon$ is independent of ε, therefore we must have the same link when passing to the limit $\varepsilon\to 0$.

Passing the Wigner Transforms to the limit

Following the ideas of [12, Theorem IV.1.] and [10, Proposition 3.5.], we would like to have for $\tilde{W}_\varepsilon$ (up to extracting a subsequence) a uniform convergence on every compact of $[0,\infty)$ in $\mathcal{A}'_{w-*}$ to a limit $\tilde{W}\in\mathcal{C}_b([0,\infty),\mathcal{M})$ thanks to Ascoli's theorem. But we cannot prove the equicontinuity of $\langle a, W_\varepsilon\rangle$ for every $a\in\mathcal{A}$ in the same way due to the very little regularity of the "potential" $V_\varepsilon = \ln|u_\varepsilon|^2$. However, we still have uniform (both in time and ε) bounds of $\tilde{W}_\varepsilon(t)$ in $\mathcal{A}'$ and of $\tilde{W}^H_\varepsilon(t)$ in $L^1(\mathbb{R}^d\times\mathbb{R}^d)$. Such a property will allow us to have some weak convergence up to a subsequence, as stated in the next lemma:

Lemma 2.6. *Under the assumptions of Theorem 1.5, there exists two (non-negative) finite measures W and $\tilde{W}$ in $L^\infty((0,\infty),\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d))$ such that, up to a subsequence, for every $p\in(1,\infty)$*

$$\begin{aligned}
W_\varepsilon &\underset{n\to\infty}{\rightharpoonup} W && \textit{in } L^p_{loc}((0,\infty),\mathcal{A}'),\\
\tilde{W}_\varepsilon &\underset{n\to\infty}{\rightharpoonup} \tilde{W} && \textit{in } L^p_{loc}((0,\infty),\mathcal{A}'),\\
W^H_\varepsilon &\underset{n\to\infty}{\rightharpoonup} W && \textit{in } L^p_{loc}((0,\infty),\mathcal{M}(\mathbb{R}^d)),\\
\tilde{W}^H_\varepsilon &\underset{n\to\infty}{\rightharpoonup} \tilde{W} && \textit{in } L^p_{loc}((0,\infty),\mathcal{M}(\mathbb{R}^d)),
\end{aligned}$$

Proof. We will focus on $\tilde{W}_\varepsilon$ since the same argument will work for W_ε.

As previously said, $\tilde{W}_\varepsilon$ and $\tilde{W}^H_\varepsilon$ are bounded respectively in $L^\infty((0,\infty),\mathcal{A}')$ and in $L^\infty((0,\infty),L^1(\mathbb{R}^d\times\mathbb{R}^d))$. Therefore, for every $T>0$, we can extract a subsequence ε^T_k such that $\tilde{W}_{\varepsilon^T_k}$ (resp. $\tilde{W}^H_{\varepsilon^T_k}$) weakly converges in $L^p(0,T;\mathcal{A}'_{w-*})$ (resp. $L^p(0,T;\mathcal{M})$) for any $p\in[1,\infty)$ to a limit $\tilde{W}^T\in L^\infty(0,T;\mathcal{A}')$ (resp. $\tilde{W}^T_H\in L^\infty(0,T;\mathcal{M})$).

Following the idea of Theorem 2.2, we should be able to prove that $\tilde{W}^T=\tilde{W}^T_H\in L^\infty(0,T;\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d))$. As we have

$$\tilde{W}^H_\varepsilon=\tilde{W}_\varepsilon * G_\varepsilon,\qquad \text{where } G_\varepsilon=\frac{1}{(\pi\varepsilon)^d}e^{-\frac{|x|^2+|\xi|^2}{\varepsilon}},$$

it is enough to prove that, for example, for any $\phi\in L^2(0,T;\mathcal{A})$ (or in a dense subspace of this space), $\phi * G_\varepsilon$ converges in $L^2(0,T;\mathcal{A})$ to ϕ. Knowing that

$$\mathcal{F}_\xi(\phi * G_\varepsilon)(t,x,z)=\left[\mathcal{F}_\xi\phi(t,x,z) *_x \frac{1}{(\pi\varepsilon)^{\frac{d}{2}}}e^{-\frac{|x|^2}{\varepsilon}}\right]e^{-\varepsilon\frac{|z|^2}{4}},$$

we see that for a.e. $t\in(0,T)$,

$$\begin{aligned}
||\phi(t)*G_\varepsilon-\phi(t)||_{\mathcal{A}} \le \int_{\mathbb{R}^d}\sup_x\left|\mathcal{F}_\xi\phi(t)-\mathcal{F}_\xi\phi(t) *_x \frac{1}{(\pi\varepsilon)^{\frac{d}{2}}}e^{-\frac{|x|^2}{\varepsilon}}\right|dz&\\
+\int_{\mathbb{R}^d}(1-e^{-\varepsilon\frac{|z|^2}{4}})\sup_x|\mathcal{F}_\xi\phi(t)|\,dz.&
\end{aligned}$$

Again for a.e. $t\in(0,T)$, the second term goes to 0 when ε goes to 0, and so does the first term if $\phi(t)\in\mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d)$ and so $\mathcal{F}_\xi\phi(t)\in\mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d)$. Moreover,

$$\begin{aligned}
||\mathcal{F}_\xi(\phi * G_\varepsilon)(t)||_{L^1_zL^\infty_x} &= \left|\left|\left[\mathcal{F}_\xi\phi(t) *_x \frac{1}{(\pi\varepsilon)^{\frac{d}{2}}}e^{-\frac{|x|^2}{\varepsilon}}\right]e^{-\varepsilon\frac{|z|^2}{4}}\right|\right|_{L^1_zL^\infty_x}\\
&\le \left|\left|\left|\left|\mathcal{F}_\xi\phi(t) *_x \frac{1}{(\pi\varepsilon)^{\frac{d}{2}}}e^{-\frac{|x|^2}{\varepsilon}}\right|\right|_{L^\infty_x} e^{-\varepsilon\frac{|z|^2}{4}}\right|\right|_{L^1_z}\\
&\le ||\mathcal{F}_\xi\phi(t)||_{L^1_zL^\infty_x}=||\phi(t)||_{\mathcal{A}},
\end{aligned}$$

which yields

$$||\phi(t)*G_\varepsilon-\phi(t)||_{\mathcal{A}}\le ||\phi(t)*G_\varepsilon||_{\mathcal{A}}+||\phi(t)||_{\mathcal{A}}\le 2||\phi(t)||_{\mathcal{A}}.$$

Then, for $\phi\in\mathcal{C}^\infty_c((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ for example, dominated convergence theorem shows that

$$\int_0^T ||\phi(t)*G_\varepsilon-\phi(t)||^2_{\mathcal{A}}dt \underset{\varepsilon\to0}{\longrightarrow} 0$$

which is what we wanted. Therefore, $\tilde{W}^T = \tilde{W}^T_H$. Using a diagonal extraction, we then have a limit $\tilde{W} \in L^\infty((0,\infty), \mathcal{A}' \cap \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d))$ such that (up to a subsequence) $\tilde{W}_\varepsilon$ converges to $\tilde{W}$ in $L^p_{loc}((0,\infty), \mathcal{A}'_{w-*})$. $\square$

Remark 2.13. The previous proof is actually a simple re-writing of the proof of the first part of Theorem 2.2 which is found in [12], with in addition the time-dependence.

Now that we have a limit, the relation (1.20) passes to the limit:

$$W(t,x,\xi) = \frac{||\rho_{in}||_{L^1}}{||\gamma^2||_{L^1}} \tilde{W}\left(t, \frac{x}{\tau(t)}, \tau(t)\,\xi - \dot{\tau}(t)\,x\right).$$

Lemma 2.7. *With the same notation as the previous Lemma and for the same subsequence in ε as for the convergence of $\tilde{W}_\varepsilon$, there holds*

$$|v_\varepsilon|^2 \underset{\varepsilon\to 0}{\rightharpoonup} \int_{\mathbb{R}^d} \tilde{W}(t,y,d\eta) = \tilde{\rho} \qquad \text{in } L^p_{loc}((0,\infty), L^1(\mathbb{R}^d)) \tag{2.26}$$

for every $p \in [1,\infty)$, and

$$\tilde{\rho} \in L^\infty((0,\infty), L^1_2 \cap L\log L) \cap \mathcal{C}([0,\infty), W^{-1,1} \cap L^1_w). \tag{2.27}$$

Proof. The previous property (1.13) shows that (v_ε) is ε-oscillatory uniformly on every compact $[0,T]$ in time, but also compact at infinity uniformly in time. Moreover, along with the de la Vallée-Poussin and Dunford-Pettis theorems, those properties also yield that for every $T > 0$, up to a further subsequence (depending on T), $\rho_\varepsilon = |v_\varepsilon|^2$ converges to a limit $\tilde{\rho}^T \in L^\infty((0,T), L^1_2 \cap L\log L)$ for the weak topology $\sigma\left(L^1((0,T)\times\mathbb{R}^d_x), L^\infty((0,T)\times\mathbb{R}^d_x)\right)$. Then

$$|v_\varepsilon|^2 \underset{\varepsilon\to 0}{\rightharpoonup} \tilde{\rho}^T \qquad \text{in } L^p([0,T], L^1_y)$$

for every $p \in [1,\infty)$. Using again a diagonal extraction, we obtain a limit $\tilde{\rho} \in L^\infty((0,\infty), L^1_2 \cap L\log L)$ such that $|v_\varepsilon|^2 \underset{\varepsilon\to 0}{\rightharpoonup} \tilde{\rho}$ in $L^p_{loc}([0,\infty), L^1_y)$ for every $p \in [1,\infty)$.

Then, in the same way as before, the second part of Theorem 2.2 can be generalized by adding time-dependence. Therefore, we have the following properties for a.e. t, y:

$$\int_{\mathbb{R}^d} \tilde{W}(t,y,d\eta) = \tilde{\rho}(t,y),$$
$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \tilde{W}(t,dy,d\eta) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} |v_\varepsilon(t,y)|^2\,dx = ||\gamma||^2_{L^2}.$$

The first equation shows that $\tilde{\rho}$ is totally determined by $\tilde{W}$, so there's no need of further extraction to make $|v_\varepsilon|^2$ weakly convergent: as soon as $(\tilde{W}_{\varepsilon_k})_k$ weakly converges to $\tilde{W}$, $|v_{\varepsilon_k}|^2$ weakly converges to $\tilde{\rho}$.

It remains to prove that $\tilde{\rho} \in \mathcal{C}([0,\infty), W^{-1,1} \cap L^1_w)$. Come back to the equation for $\partial_t\rho_\varepsilon$ in (2.2) where $\rho_\varepsilon = |v_\varepsilon|^2$:

$$\partial_t\rho_\varepsilon + \frac{1}{\tau^2(t)}\nabla\cdot J_\varepsilon = 0 \qquad \text{in } \mathcal{D}',$$

where we recall $J_\varepsilon = \mathrm{Im}(\varepsilon\,\overline{v_\varepsilon}\nabla v_\varepsilon)$. We also recall that $\frac{1}{\tau(t)}J_\varepsilon$ is bounded in $L^\infty((0,\infty), L^1(\mathbb{R}^d))$ uniformly in $\varepsilon > 0$. Therefore, ρ_ε is bounded in $W^{1,\infty}((0,\infty), W^{-1,1}(\mathbb{R}^d))$ uniformly in $\varepsilon > 0$. Such a bound leads to the estimation

$$||\rho_\varepsilon(t_0 + .) - \rho_\varepsilon||_{L^\infty_t W^{-1,1}_y} \le C|t_0| \qquad \text{for all } t_0 \ge 0.$$

This estimate passes to the weak limit, and therefore we get

$$||\tilde{\rho}(t_0 + .) - \tilde{\rho}||_{L^\infty_t W^{-1,1}_y} \leq C|t_0| \qquad \text{for all } t_0 \geq 0.$$

Such an estimate leads to $\tilde{\rho} \in \mathcal{C}(\mathbb{R}^+, W^{-1,1}(\mathbb{R}^d))$, and along with (1.13), we infer that $\tilde{\rho} \in \mathcal{C}(\mathbb{R}^+, L^1_w(\mathbb{R}^d))$.

□

Remark 2.14. This proves (1.21) and (1.22). The same kind of properties as (2.26) and (2.27) hold for W.

Second momentum of the Wigner Measure

We now use the bounds uniform in ε found in (1.13) and (1.14) to obtain some momentum for the Wigner measure, using also Property 2.2. For example, for the second momentum in ξ, we have:

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} |\eta|^2 \, \tilde{W}^H_\varepsilon(t, y, \eta) \, dy d\eta \leq \varepsilon^2 \, ||\nabla v_\varepsilon(t)||^2_{L^2} + \frac{\varepsilon d}{2} ||v_\varepsilon||^2_{L^2}.$$

We know that $\tilde{W}^H_\varepsilon$ weakly converges to $\tilde{W}$ in $L^p_{\text{loc}}([0,\infty), \mathcal{M})$ for every $p \in [1, \infty)$, therefore we get

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} |\eta|^2 \, \tilde{W}(t, dy, d\eta) \leq C \, \tau^2(t) \qquad \text{for a.e. } t,$$
$$\int_0^\infty \frac{\dot{\tau}(t)}{\tau(t)^3} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\eta|^2 \, \tilde{W}(t, dy, d\eta) \, dt \leq C.$$

In the same way, we also have some bounds for the second moment in y of $\tilde{W}$ uniformly in ε and t by using equality (2.23):

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} |y|^2 \, \tilde{W}(t, dy, d\eta) \leq C \qquad \text{for a.e. } t.$$

Convergence of some quantities

Proposition 2.7. *Under the assumptions of Theorem 1.5, there holds*

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \end{pmatrix} \tilde{\rho}(t, y) \, dy \underset{t \to \infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \end{pmatrix} \gamma^2(y) \, dy.$$

Proof. The first convergence is obvious, as it is constant. For the second convergence, using Lemma 2.2 and its notations, we get

$$I^\varepsilon_2(t) = \frac{1}{\tau(t)}(C_1 t + C_2).$$

Moreover, thanks to the bounds (1.13) and in particular the fact that $|y| \, v_\varepsilon$ is bounded in L^2 uniformly in t and ε, we also know that $I^\varepsilon_2(t) \underset{\varepsilon \to 0}{\longrightarrow} \int_{\mathbb{R}^d} y \, \rho(t, y) \, dy =: I_2(t)$. Therefore,

$$I_2(t) = \frac{1}{\tau(t)}(-I_1(0) + t \, I_2(0)) \underset{t \to \infty}{\longrightarrow} 0 = \int_{\mathbb{R}^d} y \, \gamma^2(y) \, dy.$$

□

Equation on $\tilde{\rho}$

Recall the definition of ρ_ε and J_ε

$$\rho_\varepsilon = |v_\varepsilon|^2,$$
$$J_\varepsilon = \mathrm{Im}(\varepsilon\, \overline{v_\varepsilon}\, \nabla v_\varepsilon),$$

with ρ_ε weakly converging to $\tilde{\rho}$ and J_ε bounded in L^1 locally uniformly in time, uniformly in ε, thanks to the properties (1.13). Then we also recall the three equations found in (2.2)-(2.4),

$$\partial_t \rho_\varepsilon + \frac{1}{\tau^2(t)} \nabla \cdot J_\varepsilon = 0,$$
$$\partial_t J_\varepsilon + \lambda\, \nabla \rho_\varepsilon + 2\lambda\, y\, \rho_\varepsilon = \frac{\varepsilon^2}{4\,\tau^2(t)} \Delta \nabla \rho - \frac{\varepsilon^2}{\tau^2(t)} \nabla \cdot (\mathrm{Re}(\nabla v_\varepsilon \otimes \overline{\nabla v_\varepsilon})) \qquad \text{in } \mathcal{D}',$$
$$\partial_t(\tau^2\, \partial_t \rho_\varepsilon) = \lambda\, (\Delta \tilde{\rho}_\varepsilon + 2\, \nabla \cdot (y\, \tilde{\rho}_\varepsilon)) - \frac{\varepsilon^2}{4\,\tau^2(t)} \Delta^2 \rho_\varepsilon + \frac{\varepsilon^2}{\tau^2(t)} \nabla \cdot \big(\nabla \cdot (\mathrm{Re}(\nabla v_\varepsilon \otimes \overline{\nabla v_\varepsilon}))\big)\,.$$

In the last equation, the weak convergence of ρ_ε yields the weak convergence of the left-hand side and of the first term of the right-hand side. Moreover, the third term of the right-hand side goes to zero in $W^{-4,1}$. Then, thanks to (1.13), the boundedness of $|y|^2 \rho_\varepsilon$ in L^1 norm leads to the weak convergence of $y\, \rho_\varepsilon$ to $y\, \tilde{\rho}$. It only remains the convergence of the last term of the right-hand side. The only bounds we have on this term are due to the same property (1.13) and the following one (1.14):

$$\left|\left| \frac{\varepsilon^2}{\tau^2(t)}\, \mathrm{Re}(\nabla v_\varepsilon(t) \otimes \overline{\nabla v_\varepsilon(t)}) \right|\right|_{L^1} \leq \frac{\varepsilon^2}{\tau^2(t)} ||\nabla v_\varepsilon(t)||^2_{L^2} \leq C \qquad \text{for every } t \geq 0 \text{ and } \varepsilon > 0,$$
$$\int_0^\infty \frac{\dot{\tau}(t)}{\tau(t)} \int_{\mathbb{R}^d} \left| \frac{\varepsilon^2}{\tau^2(t)}\, \mathrm{Re}(\nabla v_\varepsilon(t) \otimes \overline{\nabla v_\varepsilon(t)}) \right| dy\, dt \leq C.$$

Therefore, up to a subsequence, $\frac{\varepsilon^2}{\tau^2(t)} \mathrm{Re}(\nabla v_\varepsilon(t) \otimes \overline{\nabla v_\varepsilon(t)})$ weakly converges as a measure in every $[0,T] \times \mathbb{R}^d$ (T > 0) to a $\nu \in L^\infty((0,\infty), M_d(\mathcal{M}_s(\mathbb{R}^d)))$ (where $\mathcal{M}_s$ designed signed measure) which verifies

$$\int_0^\infty \frac{\dot{\tau}(t)}{\tau(t)}\, |\nu|(t, \mathbb{R}^d)\, dt \leq C < \infty. \tag{2.28}$$

Therefore, we obtain $\frac{\varepsilon^2}{\tau^2(t)} \nabla \cdot \big(\nabla \cdot (\mathrm{Re}(\nabla v_\varepsilon \otimes \overline{\nabla v_\varepsilon}))\big) \underset{\varepsilon \to 0}{\rightharpoonup} \nabla \cdot (\nabla \cdot \nu)$ in the sense that for every $\phi \in \mathcal{C}_c((0,\infty); \mathcal{C}_0^2(\mathbb{R}^d))$,

$$\left\langle \frac{\varepsilon^2}{\tau^2(t)} \nabla \cdot \big(\nabla \cdot (\mathrm{Re}(\nabla v_\varepsilon \otimes \overline{\nabla v_\varepsilon}))\big)\,, \phi \right\rangle_{t,y} = \left\langle \frac{\varepsilon^2}{\tau^2(t)} (\mathrm{Re}(\nabla v_\varepsilon \otimes \overline{\nabla v_\varepsilon})), \nabla \otimes \nabla \phi \right\rangle_{t,y}$$

where the last matrix scalar product must be interpreted term by term: $\langle A, B \rangle = \sum_{i,j} a_{ij} b_{ij}$, and so

$$\left\langle \frac{\varepsilon^2}{\tau^2(t)} \nabla \cdot \big(\nabla \cdot (\mathrm{Re}(\nabla v_\varepsilon \otimes \overline{\nabla v_\varepsilon}))\big)\,, \phi \right\rangle_{t,y} \underset{\varepsilon \to 0}{\longrightarrow} \langle \nu, \nabla \otimes \nabla \phi \rangle_{t,y} = \langle \nabla \cdot \nabla \cdot \nu, \phi \rangle_{t,y}.$$

Thus

$$\partial_t(\tau^2\, \partial_t \tilde{\rho}) = \lambda\, L \tilde{\rho} + \nabla \cdot (\nabla \cdot \nu)\,, \tag{2.29}$$

where we recall $L = \Delta + \nabla \cdot (2y\, .)$ is the same Fokker-Plank operator as previously.

In order to reproduce the same proof as previously, we want a similar equation as (2.2). The first term still converges, and the fact that J_ε has some good bounds is helpful. Recalling such bounds,

$$\frac{1}{\tau(t)}||J_\varepsilon(t)||_{L^1} \leq C,$$
$$\int_{\mathbb{R}^d} \frac{1}{\tau(t)}|J_\varepsilon(t,y)|\,|y|\,dy \leq C,$$
$$\int_0^\infty \frac{\dot\tau(t)}{\tau(t)}\left(\frac{1}{\tau(t)}||J_\varepsilon(t)||_{L^1}\right)^2 dt \leq C,$$

the first two inequalities imply that $\frac{1}{\tau(t)}J_\varepsilon$ is a tight sequence uniformly in ε and time. Therefore, up to a further subsequence (in ε), $\frac{1}{\tau(t)}J_\varepsilon$ narrowly converges in measure in $[0,T]\times\mathbb{R}^d$ for every $T>0$ to a limit $\mu \in L^\infty([0,\infty),\mathcal{M}_s(\mathbb{R}^d)^d)$ with also for every $t \geq 0$

$$\int_{\mathbb{R}^d} |y|\,d|\mu|(t) \leq C, \tag{2.30}$$
$$\int_0^\infty \frac{\dot\tau(t)}{\tau(t)}\,|\mu|(t,\mathbb{R}^d)^2\,dt \leq C, \tag{2.31}$$

and the weak limit of equation (2.2)

$$\partial_t\tilde\rho + \frac{1}{\tau(t)}\nabla\cdot\mu = 0. \tag{2.32}$$

ACTUALLY, μ IS CONTINUOUS IN TIME THANKS TO (2.4) IN S VARIABLE AND (2.30), AND WE WILL NEED IT !!!!!!!!!!!!

End of the proof

Using the 5 equations and estimates (2.28)-(2.32), the proof is then similar to the previous proof presented in Section 2.1.2. Indeed, the same considerations hold, except that we need to take bounded continuous functions instead of L^∞ (or C_b^n when we talked about $W^{n,\infty}$ for $n\in\mathbb{N}$) because $\frac{\varepsilon^2}{\tau^2(t)}\operatorname{Re}(\nabla v_\varepsilon\otimes\overline{\nabla v_\varepsilon})$ and $\frac{1}{\tau(t)}J_\varepsilon$, both $L^\infty((0,\infty),L^1(\mathbb{R}^d))$, are replaced respectively by $\nu\in L^\infty((0,\infty),M_d(\mathcal{M}_s(\mathbb{R}^d)))$ and $\mu\in L^\infty((0,\infty),\mathcal{M}_s(\mathbb{R}^d)^d)$, and thus $\langle\mu(t),\phi\rangle_y$ (for example) has a meaning for a.e. $t>0$ if $\phi\in\mathcal{C}_b(\mathbb{R}^d)$ instead of $L^\infty(\mathbb{R}^d)$. However, the estimations are totally similar and the proof holds once we have made this little modification.

For example, using the two same notations as in the previous proof, we have $\nabla\cdot(\nabla\cdot\check\nu_n)\rightharpoonup 0$ in the sense that for every $\psi\in C_b^2([-1,\infty)\times\mathbb{R}^d)$

$$\begin{aligned}\langle\nabla\cdot(\nabla\cdot\check\nu_n),\psi\rangle &= \iint_{[-1,\infty)\times\mathbb{R}^d} D^2\psi\,d\check\nu_n\\ |\langle\nabla\cdot(\nabla\cdot\check\nu_n),\psi\rangle| &\leq ||D^2\phi||_{C^0}\iint_{[-1,\infty)\times\mathbb{R}^d} d|\check\nu_n|\\ &\leq ||D^2\phi||_{C^0}\iint_{[s_{\phi(n)-1},\infty)\times\mathbb{R}^d} d|\check\nu| \underset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

thanks to (2.28) transformed in a property in s.

Remark 2.15. We insist on the fact that we can consider $\tilde\rho(t)$ for *all* $t\geq 0$ (and not only a.e.) thanks to the property $\tilde\rho\in\mathcal{C}(\mathbb{R}^+,L^1_w)$. We also remark that the condition (2.10) still holds thanks to this property, which makes the proof hold in the same way.

□

Remark 2.16. In some way, this theorem shows that the weak convergence of $\rho_\varepsilon = |v_\varepsilon|^2$ to γ^2 is uniform in $\varepsilon \in (0,1]$, since we have the same convergence for $\varepsilon = 0$. This uniformity might be quantified, for example in some Wasserstein distance, but we will not do it in this survey. It will probably be the subject of a further work.

Remark 2.17. Thanks to the equality (2.21), we see that the limit μ of $\frac{1}{\tau(t)}\tilde{J}_\varepsilon$ is actually

$$\mu = \frac{1}{\tau(t)} \int_{\mathbb{R}^d} \eta \, \tilde{W}(t, y, d\eta).$$

The same consideration would have worked for ν with "$\nu = \frac{1}{\tau^2(t)} \int_{\mathbb{R}^d} \eta \otimes \eta \, \tilde{W}(t, y, d\eta)$" if we had a bound for a higher momentum in η for example.

2.3 Vlasov Equation with Logarithmic Non-Linearity

We recall the Vlasov Equation with Logarithmic Non-Linearity (1.1):

$$\partial_t f + \xi \cdot \nabla_x f - \lambda \, \nabla_x (\ln \rho) \cdot \nabla_\xi f = 0,$$

where $\lambda > 0$ and

$$\rho(t,x) = \int f(t,x,d\xi).$$

A solution $f = f(t,x,\xi)$ of such a Vlasov equation is a non-negative measure in x and ξ for every t.

We explain first, in a very formal way, how we end up with the assumptions (1.30)-(1.35). Recall the first two assumptions (1.30) and (1.31):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} f(t, dx, d\xi) \right) = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\xi|^2 \, f(t, dx, d\xi) + \lambda \int_{\mathbb{R}^d} \rho(t,x) \ln \rho(t,x) dx \right) = 0.$$

The first equality comes from the fact that this type of Vlasov equation is a transport equation with null-divergence transport. The second equality is a well-known property of energy, since the potential in this case is $\ln \rho$. Therefore, if we want such properties, we will need all the terms to be well-defined, especially in (1.31), for every t, and thus also for the initial data, even in the case where $f = f(t,x,\xi)$ is only a measure in x and ξ for every t. The term $\rho \ln \rho$ is the most interesting, as it cannot be generalized to the case where ρ is a measure. Therefore, we need $\rho(t,x)$ to be defined x-a.e. for every t, and then $\rho(t) \in L^1(\mathbb{R}^d)$ with (1.30). Moreover, in the same way as previously, we also have some other (formal) properties, for example for ρ:

$$\partial_t \rho(t,x) + \nabla_x \cdot \left(\int_{\mathbb{R}^d} \xi \, f(t,x,d\xi) \right) = 0,$$

or also for $\int_{\mathbb{R}^d} \xi \, f(t,x,d\xi)$:

$$\partial_t \int_{\mathbb{R}^d} \xi \, f(t,x,d\xi) + \nabla_x \cdot \int_{\mathbb{R}^d} \xi \otimes \xi \, f(t,x,d\xi) + \lambda \, \nabla_x \rho(t,x) = 0,$$

These two equations yield for example

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x|^2\, f(t,dx,d\xi)\right) = 2\iint_{\mathbb{R}^d\times\mathbb{R}^d} x\cdot\xi\, f(t,dx,d\xi),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} x\cdot\xi\, f(t,dx,d\xi)\right) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\xi|^2\, f(t,dx,d\xi) + \lambda\int_{\mathbb{R}^d}\rho(t,x)dx,$$

and thus we also want those terms to be well-defined. Such remarks lead us to define:

$$\mathcal{M}\Sigma_{\log} = \left\{\mu\in\mathcal{M}(\mathbb{R}^d_x\times\mathbb{R}^d_\xi), \rho(x) = \int_{\mathbb{R}^d_\xi}\mu(x,d\xi)\in L^1(\mathbb{R}^d)\cap L\,|\log L\,|(\mathbb{R}^d)\right\},$$
$$\mathcal{M}_2 = \left\{\mu\in\mathcal{M}(\mathbb{R}^d_x\times\mathbb{R}^d_\xi), \iint_{\mathbb{R}^d\times\mathbb{R}^d}(|x|^2+|\xi|^2)\,d\mu<\infty\right\}.$$

Remark 2.18. $\nabla_x(\ln\rho(t))$ is actually weakly defined $\rho(t)$-a.e.: indeed, for every $\phi\in W^{1,\infty}(\mathbb{R}^d)$,

$$\int\nabla_x(\ln\rho)(t,.)\,\phi\,d\rho(t) = -\int\rho(t,x)\,\nabla\phi(x)\,dx = -\int\nabla\phi\,d\rho(t).$$

In the same way, the term $\nabla_x(\ln\rho)\cdot\nabla_\xi f$ is weakly well-defined as soon as $\rho(t)\in W^{1,1}$ because for every $\phi\in L^\infty(\mathbb{R}^d_x, W^{1,\infty}(\mathbb{R}^d_\xi))$

$$\begin{aligned}\langle\nabla_x(\ln\rho)(t,x)\cdot\nabla_\xi f(t,x,\xi),\phi(x,\xi)\rangle_{(x,\xi)} &= \langle\nabla_x(\ln\rho)(t,x)\,f(t,x,\xi),\nabla_\xi\phi(x,\xi)\rangle_{(x,\xi)}\\ &= \Big\langle\nabla_x(\ln\rho)(t,x),\langle f(t,x,\xi),\nabla_\xi\phi(x,\xi)\rangle_\xi\Big\rangle_x,\end{aligned}$$

with the last term well-defined because:

$$\begin{aligned}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\nabla_x(\ln\rho)(t,x)\,f(t,x,\xi)\cdot\nabla_\xi\phi(x,\xi)|\;dxd\xi &\le \int_{\mathbb{R}^d_x}|\nabla_x(\ln\rho)(t,x)|\int_{\mathbb{R}^d_\xi}|f(t,x,\xi)\,\nabla_\xi\phi(x,\xi)|\;d\xi\,dx\\ &\le \int_{\mathbb{R}^d_x}|\nabla_x(\ln\rho)(t,x)|\int_{\mathbb{R}^d_\xi}f(t,x,\xi)\,||\nabla_\xi\phi||_{L^\infty}\;d\xi\,dx\\ &\le \int_{\mathbb{R}^d_x}|\nabla_x(\ln\rho)(t,x)|\,\rho(t,x)\,||\nabla_\xi\phi||_{L^\infty}\;dx\\ &\le \int_{\mathbb{R}^d_x}|\nabla_x\rho(t,x)|\;||\nabla_\xi\phi||_{L^\infty}\;dx\;<\infty.\end{aligned}$$

Such remarks might help in order to find a real formalization of the equation, but this is not our goal here.

2.3.1 "Gaussian-Gaussian" Case in dimension 1

The "Gaussian-Gaussian" case is very interesting because the solutions can actually be computed explicitly. It is also helpful to understand the behaviour of the solution, and to see that the dispersion rate is the same in this case as the previous one in $t\sqrt{\ln t}$.

Proposition 2.8. *1. For $c_{1,0}$, $c_{2,0}>0$, $c_{1,1}, B_0, B_1\in\mathbb{R}$, set*

$$\tilde{C} = c_{1,0}\,c_{2,0}, \tag{2.33}$$

and consider $c_1 \in \mathcal{C}^\infty(\mathbb{R}^+)$ *solution to*

$$\begin{cases} \ddot{c}_1 = \dfrac{2\lambda}{c_1} + \dfrac{\tilde{C}^2}{c_1^3}, \\ c_1(0) = c_{1,0}, \\ \dot{c}_1(0) = c_{1,1}. \end{cases} \tag{2.34}$$

Then set

$$c_2(t) = \frac{\tilde{C}}{c_1(t)}, \tag{2.35}$$

$$b_1(t) = B_1 t + B_0, \tag{2.36}$$

$$b_2(t,x) = \frac{\dot{c}_1(t)}{c_1(t)}(x - B_1\, t - B_0) + B_1. \tag{2.37}$$

Therefore

$$f(t,x,\xi) = \frac{1}{\pi\, c_1(t)\, c_2(t)}\ \exp\left[-\frac{|x-b_1(t)|^2}{c_1(t)^2} - \frac{|\xi - b_2(t,x)|^2}{c_2(t)^2}\right] \tag{2.38}$$

satisfies (1.1).

2. *Let* $T \in (0,\infty) \cup \{+\infty\}$, $b_1 = b_1(t) \in \mathcal{C}^1([0,T),\mathbb{R})$, $c_1 = c_1(t) \in \mathcal{C}^1([0,T),(0,\infty)) \in$, $b_2 = b_2(t,x) \in \mathcal{C}^1([0,T) \times \mathbb{R}, \mathbb{R})$ *and* $c_2 = c_2(t,x) \in \mathcal{C}^1([0,T) \times \mathbb{R}, (0,\infty))$ *such that*

$$f(t,x,\xi) = \frac{1}{\pi\, c_1(t)\, c_2(t,x)}\ \exp\left[-\frac{|x-b_1(t)|^2}{c_1(t)^2} - \frac{|\xi - b_2(t,x)|^2}{c_2(t,x)^2}\right] \tag{2.39}$$

is a solution of (1.1). *Then* $c_2 = c_2(t)$, *all the functions are* $\mathcal{C}^2$ *and* (2.33)-(2.37) *hold.*

The proof of this Proposition actually needs a lot of computations. This will be done in Appendix C.

Remark 2.19. This property may also handle the case when $c_{2,0} = 0$, which is actually the monokinetic case where we have a Dirac in ξ:

$$f_{in}(x,\xi) = \frac{1}{\sqrt{\pi}\, c_{1,0}}\ \exp\left[-\frac{(x-b_{1,0})^2}{c_{1,0}^2}\right] \otimes \delta_{\xi = b_{2,0}(x)}.$$

where $b_{2,0}(x)$ is affine. Then the previous proposition shows that f is a Dirac in ξ for all time (if we only consider Gaussian solutions), as $c_1(t)\, c_2(t) = c_1(0)\, c_2(0) = 0$ with $c_1(t) \neq 0\ \forall t$. This is similar to [7].

Remark 2.20. The behaviour of c_1 has already been studied in [7]:

$$c_1(t) \underset{t\to\infty}{\sim} 2t\sqrt{\lambda \ln t}.$$

We see that this is the same dispersion rate as in [7] or [5], which is good news. It allows us to get some convergence, as stated in the following corollary.

Corollary 2.2. *With the notations and assumptions of Proposition 2.8, if we rescale to* $\tilde{f} = \tilde{f}(t,y,\eta)$ *like previously*

$$f(t,x,\xi) = \frac{||\rho_{in}||_{L^1}}{||\gamma^2||_{L^1}}\, \tilde{\mu}\left(t, \frac{x}{\tau(t)}, \tau(t)\,\xi - \dot{\tau}(t)\, x\right),$$

and define

$$\tilde{\rho}(t,y) := \int_{\mathbb{R}^d} f(t,y,\eta)\, d\eta,$$

there holds

$$\tilde{\rho}(t,.) \underset{t\to\infty}{\longrightarrow} \gamma^2 \qquad in\ L^1(\mathbb{R}).$$

2.3.2 General Case

Change of variables

Recall the change of variables (1.36) and write it in terms of $\tilde{f}$ and $\tilde{\rho}$

$$f(t,x,\xi) = \frac{M}{||\gamma^2||_{L_1}} \tilde{f}\left(t, \frac{x}{\tau(t)}, \tau(t)\xi - \dot{\tau}(t)x\right)$$
$$\iff \tilde{f}(t,y,\eta) = \frac{||\gamma^2||_{L_1}}{M} f\left(t, \tau(t)y, \frac{\eta}{\tau(t)} + \dot{\tau}(t)y\right),$$
$$\tilde{\rho}(t,y) = \frac{||\gamma^2||_{L_1}}{M} \rho(t, \tau(t)y)\, \tau(t)^d.$$

Equation on $\tilde{f}$

The Vlasov equation (1.1) can then be expressed in terms of the previous change of variables. Computation yields:

$$\partial_t \tilde{f} - \lambda\left[2y + \nabla_y(\ln\tilde{\rho})\right]\cdot\nabla_\eta \tilde{f} + \frac{\eta}{\tau(t)^2}\nabla_y \tilde{f} = 0.$$

Energy estimates

We define the relative entropy, the (modified) kinetic energy and the total modified energy by:

$$\mathcal{E}_{\text{ent}}(t) := \int_{\mathbb{R}^d} \tilde{\rho}(t,y)\ln\left(\frac{\tilde{\rho}(t,y)}{\gamma^2}\right) dy = \int_{\mathbb{R}^d} \tilde{\rho}(t,y)\left(\ln\tilde{\rho}(t,y) + |y|^2\right) dy,$$
$$\mathcal{E}_{\text{kin}}(t) := \frac{1}{2\,\tau^2(t)} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\eta|^2 \tilde{f}(t,dy,d\eta),$$
$$\mathcal{E}(t) := \mathcal{E}_{kin}(t) + \lambda\,\mathcal{E}_{ent}(t).$$

Then we easily compute thanks to assumptions (1.30), (1.31), (1.34) and (1.35):

$$\dot{\mathcal{E}} = -2\frac{\dot{\tau}(t)}{\tau(t)}\mathcal{E}_{\text{kin}}.$$

Therefore, in the same way as previously:

Proposition 2.9. *Under the assumptions of Theorem 1.7, there holds:*

$$\sup_{t\geq 0}\left(\int_{\mathbf{R}^d}\left(|\ln\tilde{\rho}(t,y)| + |y|^2\right)\tilde{\rho}(t,y)\,dy + \frac{1}{\tau^2(t)}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\eta|^2\tilde{f}(t,dy,d\eta)\right) < \infty, \tag{2.40}$$

and

$$\int_0^\infty \frac{\dot{\tau}(t)}{\tau^3(t)}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\eta|^2\tilde{f}(t,dy,d\eta)\,dt < \infty. \tag{2.41}$$

Proof. The proof is totally similar to the proof of Lemma 2.1. □

Remark 2.21. Again, as we know that $\tilde{\rho}\geq 0$ and $||\tilde{\rho}(t)||_{L^1(\mathbb{R}^d)} = ||\gamma^2||_{L^1(\mathbb{R}^d)}$, Csiszár-Kullback inequality reads (see [1, Theorem 8.2.7]):

$$\mathcal{E}_{ent}(t) \geq \frac{1}{2\,||\gamma^2||_{L^1(\mathbb{R}^d)}}||\tilde{\rho}(t) - \gamma^2||^2_{L^1(\mathbb{R}^d)},$$

hence in particular $\mathcal{E}\geq\mathcal{E}_{ent}\geq 0$.

System of equations involving $\tilde{\rho}$

Define

$$\tilde{J}(t,y) := \int \eta \, \tilde{f}(t,y,d\eta).$$

Thanks to assumptions (1.32) and (1.33), we get the system

$$\begin{cases} \partial_t \tilde{\rho} + \dfrac{1}{\tau^2} \nabla_y \cdot \tilde{J} = 0, & (2.42) \\ \partial_t \tilde{J} + \lambda(2y\, \tilde{\rho} + \nabla_y \tilde{\rho}) = -\dfrac{1}{\tau^2} \nabla_y \left(\displaystyle\int |\eta|^2 \, \tilde{f}(t,y,d\eta) \right). & (2.43) \end{cases}$$

Convergence of some quadratic quantities

Proposition 2.10. *Under the assumptions of Theorem 1.7, there holds:*

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \tilde{\rho}(t,y)\, dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \gamma^2(y)\, dy.$$

Proof. Again, the proof is similar to Lemma 2.2.

- The first convergence is obvious thanks to the mass conservation.
- The second convergence follows from

  $$\dot{I}_2(t) = \frac{1}{\tau^2(t)} I_1(t), \qquad \dot{I}_1(t) = -2\lambda\, I_2(t),$$

  where

  $$I_1(t) = \int_{\mathbb{R}^d} \tilde{J}(t,dy), \qquad I_2(t) = \int_{\mathbb{R}^d} y\, \tilde{\rho}(t,y)\, dy,$$

  thanks to (2.42) and (2.43).
- The third convergence is proven thanks to the energy conservation, translating it into estimates on $\tilde{f}$.

□

Regularity of $\tilde{\rho}$

Come back to equation (2.42):

$$\partial_t \tilde{\rho} + \frac{1}{\tau^2} \nabla_y \cdot \tilde{J} = 0.$$

The estimate (2.40) yields in the same way as previously that $\frac{1}{\tau} \tilde{J} \in L^\infty((0,\infty), \mathcal{M}_s(\mathbb{R}^d)^d)$. In particular, the embedding $W^{2,\infty}(\mathbb{R}^d) \hookrightarrow \mathcal{C}^1(\mathbb{R}^d)$ shows that $\frac{1}{\tau^2} \nabla_y \cdot \tilde{J} \in L^\infty((0,\infty), W^{-2,1}(\mathbb{R}^d))$. Therefore, the equation (2.42) leads to

$$\tilde{\rho} \in W^{1,\infty}((0,\infty), W^{-2,1}(\mathbb{R}^d)) \subset \mathcal{C}((0,\infty), W^{-2,1}(\mathbb{R}^d)).$$

Using again the estimates (2.40), there holds therefore $\tilde{\rho} \in \mathcal{C}(\mathbb{R}^+, L^1_w(\mathbb{R}^d))$.

Remark 2.22. The same considerations hold for ρ : $\rho \in \mathcal{C}(\mathbb{R}^+, L^1_w(\mathbb{R}^d))$.

End of the proof

Thanks to the estimations (2.40) and (2.41) and the equations (2.42) and (2.43), the proof is actually completely similar to the proof for the Wigner Measure made in Section 2.2.2.

□

Remark 2.23. We can barely say anything on the behaviour of $\tilde{J}$ except some integration properties (in y and time). The only other property we might say comes from (2.43). Indeed, as we now have $\tilde{\rho}_n \rightharpoonup \gamma^2$ in $L^p_{t,loc}(0,\infty;L^1(\mathbb{R}^d))$ and since we also have property (2.40), we know that

$$2y\,\tilde{\rho}_n + \nabla_y \tilde{\rho}_n \underset{n\to\infty}{\rightharpoonup} 0 \qquad \text{in } L^1_{t,loc}(0,\infty;W^{-1,1}(\mathbb{R}^d)).$$

Moreover, (2.41) yields that for every $\phi \in L^\infty(0,\infty;C^1_b(\mathbb{R}^d))$

$$\begin{aligned}\left|\left\langle \frac{\dot{\tau}_n}{\tau_n^3}\,\nabla_y\left(\int_{\mathbb{R}^d}|\eta|^2\,\tilde{f}_n(t,y,d\eta)\right),\phi\right\rangle_{t,y}\right| &= \int_0^\infty \frac{\dot{\tau}_n(t)}{\tau_n(t)^3}\int_{\mathbb{R}^d}\nabla_y\phi(t,y)\int_{\mathbb{R}^d}|\eta|^2\,\tilde{f}_n(t,dy,d\eta)\,dt \\ &\le ||\nabla_y\phi||_\infty \int_{t_n}^\infty \frac{\dot{\tau}(t)}{\tau(t)^3}\int_{\mathbb{R}^d\times\mathbb{R}^d}|\eta|^2\,\tilde{f}(t,dy,d\eta)\,dt \\ &\underset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Therefore, as $\frac{\dot{\tau}(t)}{\tau(t)} \underset{t\to\infty}{\sim} \frac{1}{t}$, (2.43) gives that for every $\phi \in L^\infty(0,\infty;C^1_b(\mathbb{R}^d))$

$$\left\langle \frac{\dot{\tau}_n}{\tau_n}\partial_t \tilde{J}_n,\phi\right\rangle_{t,y} \underset{n\to\infty}{\longrightarrow} 0.$$

Proof of Corollary 1.1

In the energy for f, write the potential energy in terms of $\tilde{\rho}$.

$$\begin{aligned}\int_{\mathbb{R}^d}\rho(t,x)\ln\rho(t,x)\,dx &= -d\frac{M}{||\gamma^2||_{L^1}}\ln\tau(t)\int_{\mathbb{R}^d}\rho(t,y)\,dy + \frac{M}{||\gamma^2||_{L^1}}\int_{\mathbb{R}^d}\rho(t,y)\ln\rho(t,y)\,dy \\ &\qquad + \frac{M}{||\gamma^2||_{L^1}}\ln\frac{M}{||\gamma^2||_{L^1}}\int_{\mathbb{R}^d}\rho(t,y)\,dy \\ &= -dM\ln\tau(t) + \mathcal{O}(1),\end{aligned}$$

from (2.40). Therefore, the conservation of the energy for f yields

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\xi|^2\,f(t,dx,d\xi) = 2E_0 + 2\lambda dM\ln\tau(t) + \mathcal{O}(1) \underset{t\to\infty}{\sim} 2\lambda dM\ln t.$$

Conclusion and perspectives

The universal dynamics for the non-linear Schrödinger equation with logarithmic non-linearity found in [5] with no quantum semi-classical constant were generalized for the case with semi-classical constant, and propagate through the semi-classical limit via the Wigner Transform: any Wigner Measure limit of a subsequence of the Wigner Measure induces a density in space which weakly converges to the same gaussian than previously up to a rescaling, with the same dispersion rate altered by a logarithmic factor. Already proven for the case of mono-kinetic solutions, i.e. for solutions of the isothermal Euler equation, in [4], such dynamics still hold for the "solutions" of the non-linear Vlasov equation with logarithmic non-linearity, with the same logarithmic potential.

However, the previous result was proven by assuming formal properties of the Logarithmic Vlasov Equation, and a formalization of this equations still needs to be found. The formal link between Wigner Measure and the Logarithmic Vlasov Equation also needs to be proven in a more rigorous way.

Yet, we have seen that the Wigner Measure we have found is, in some way, mono-kinetic at $t = 0$. In the light of the considerations of [6], we can wonder if it remains mono-kinetic, at least for small time and for analytic initial data. The second perspective to those results lie in the convergence to γ^2 for both the solution for the Logarithmic Schrödinger Equation and the Wigner Measure. Since the convergence still holds for both case $\varepsilon > 0$ and $\varepsilon = 0$, we can wonder if such a convergence may be uniform in ε in some way. As previously said, such an uniformity might be found in the Wasserstein metric, and we need to deepen our computations to prove such a conjecture.

Appendix A

Proof of Proposition 2.2

The part 1 of this proposition can be found in [12]. We now prove the points 2 to 4. The part 2 is proven in Section A.1, Section A.2 is devoted to the proof of part 3, and finally we prove part 4 in Section A.3.

A.1 First part: proof of the second momentum in ξ

The proof of part 2 of Proposition 2.2 is organized in 4 parts. First, we will prove the equality of $\int_{\mathbb{R}^d} |\xi|^2 \, W_\varepsilon^H(x,\xi)\, d\xi$ for $u_\varepsilon \in H^2$ because we need some better regularity to prove the exchange of integral we will make. Then, we will generalize this result to the case $u_\varepsilon \in H^1$ by using an argument of continuity of such a quadratic form and the fact that the integral is still well-defined even if $u_\varepsilon \in H^1$ as $|\xi|^2 \, W_\varepsilon^H(x,\xi) \geq 0$. Then we will be able to consider $\xi_i \xi_j \, W_\varepsilon^H(x,\xi)$ without any issue, and we will prove the equality involving it in the same way: first for $u_\varepsilon \in H^2$, and then generalizing it for $u_\varepsilon \in H^1$ thanks to a continuity argument.

A.1.1 Scalar second momentum: H^2 case

As W_ε^H is non-negative, we can consider $\int_{\mathbb{R}^d} |\xi|^2 \, W_\varepsilon^H(x,\xi)\, d\xi$ without any issue. Moreover, we suppose here that $u_\varepsilon \in H^2$. Then:

$$\begin{aligned}
\int_{\mathbb{R}^d} |\xi|^2 \, W_\varepsilon^H(x,\xi)\, d\xi &= \int_{\mathbb{R}^d} |\xi|^2 \, (W_\varepsilon *_\xi \gamma_\varepsilon *_x \gamma_\varepsilon)(x,\xi)\, d\xi \\
&= \left(\int_{\mathbb{R}^d} |\xi|^2 \, W_\varepsilon *_\xi \gamma_\varepsilon \, d\xi \right) *_x \gamma_\varepsilon(x).
\end{aligned}$$

Let's check that the previous integral exchange is rigorous.

$$\begin{aligned}
&\left(\int_{\mathbb{R}^d} |\xi|^2 \; |W_\varepsilon *_\xi \gamma_\varepsilon| \; d\xi \right) *_y \gamma_\varepsilon(x) \\
&\quad = \left(\int_{\mathbb{R}^d} |\xi|^2 \left| \mathcal{F}_{z\to\xi} \left(u_\varepsilon \left(. + \frac{\varepsilon}{2} z \right) \overline{u_\varepsilon \left(. - \frac{\varepsilon}{2} z \right)} \right) *_\xi \mathcal{F}_{z\to\xi} \left(\exp \left(-\varepsilon \frac{|z|^2}{4} \right) \right) \right| d\xi \right) * \gamma_\varepsilon(x) \\
&\quad = \left(\int_{\mathbb{R}^d} \left| |\xi|^2 \, \mathcal{F}_{z\to\xi} \left(u_\varepsilon \left(. + \frac{\varepsilon}{2} z \right) \overline{u_\varepsilon \left(. - \frac{\varepsilon}{2} z \right)} \exp \left(-\varepsilon \frac{|z|^2}{4} \right) \right) \right| d\xi \right) * \gamma_\varepsilon(x) \\
&\quad = \left(\int_{\mathbb{R}^d} \left| \mathcal{F}_{z\to\xi} \left(\Delta_z \left(u_\varepsilon \left(. + \frac{\varepsilon}{2} z \right) \overline{u_\varepsilon \left(. - \frac{\varepsilon}{2} z \right)} \exp \left(-\varepsilon \frac{|z|^2}{4} \right) \right) \right) \right| d\xi \right) * \gamma_\varepsilon(x).
\end{aligned}$$

Then we have

$$
\begin{aligned}
&\Delta_z \left(u_\varepsilon \left(x + \frac{\varepsilon}{2}\, z \right) \overline{u_\varepsilon \left(x - \frac{\varepsilon}{2}\, z \right)} \exp\left(-\varepsilon \frac{|z|^2}{4} \right) \right) \\
&\qquad = \frac{\varepsilon}{2} \nabla_z \cdot \left(\nabla u_\varepsilon \left(x + \frac{\varepsilon}{2}\, z \right) \overline{u_\varepsilon \left(x - \frac{\varepsilon}{2}\, z \right)} \exp\left(-\varepsilon \frac{|z|^2}{4} \right) \right. \\
&\qquad \left. - u_\varepsilon \left(x + \frac{\varepsilon}{2}\, z \right) \overline{\nabla u_\varepsilon \left(x - \frac{\varepsilon}{2}\, z \right)} \exp\left(-\varepsilon \frac{|z|^2}{4} \right) - u_\varepsilon \left(x + \frac{\varepsilon}{2}\, z \right) \overline{u_\varepsilon \left(x - \frac{\varepsilon}{2}\, z \right)}\, z \exp\left(-\varepsilon \frac{|z|^2}{4} \right) \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\varepsilon^2}{4} \left[\Delta u_\varepsilon \left(x + \frac{\varepsilon}{2}\, z \right) \overline{u_\varepsilon \left(x - \frac{\varepsilon}{2}\, z \right)} \exp\left(-\varepsilon \frac{|z|^2}{4} \right) + u_\varepsilon \left(x + \frac{\varepsilon}{2}\, z \right) \overline{\Delta u_\varepsilon \left(x - \frac{\varepsilon}{2}\, z \right)} \exp\left(-\varepsilon \frac{|z|^2}{4} \right) \right. \\
&+ u_\varepsilon \left(x + \frac{\varepsilon}{2}\, z \right) \overline{u_\varepsilon \left(x - \frac{\varepsilon}{2}\, z \right)}\, |z|^2 \exp\left(-\varepsilon \frac{|z|^2}{4} \right) - 2 \nabla u_\varepsilon \left(x + \frac{\varepsilon}{2}\, z \right) \overline{\nabla u_\varepsilon \left(x - \frac{\varepsilon}{2}\, z \right)} \exp\left(-\varepsilon \frac{|z|^2}{4} \right) \\
&\left. - 2 \nabla u_\varepsilon \left(x + \frac{\varepsilon}{2}\, z \right) \overline{u_\varepsilon \left(x - \frac{\varepsilon}{2}\, z \right)} \cdot z \exp\left(-\varepsilon \frac{|z|^2}{4} \right) + 2\, u_\varepsilon \left(x + \frac{\varepsilon}{2}\, z \right) \overline{\nabla u_\varepsilon \left(x - \frac{\varepsilon}{2}\, z \right)} \cdot z \exp\left(-\varepsilon \frac{|z|^2}{4} \right) \right] \\
&- \frac{\varepsilon d}{2} u_\varepsilon \left(x + \frac{\varepsilon}{2}\, z \right) \overline{u_\varepsilon \left(x - \frac{\varepsilon}{2}\, z \right)} \exp\left(-\varepsilon \frac{|z|^2}{4} \right).
\end{aligned}
$$

Then, replacing $\Delta_z \left(u_\varepsilon \left(x + \frac{\varepsilon}{2}\, z \right) \overline{u_\varepsilon \left(x - \frac{\varepsilon}{2}\, z \right)} \exp\left(-\varepsilon \frac{|z|^2}{4} \right) \right)$ in the previous calculus by any of the terms of the previous sum leads to a finite number. For example:

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^d} \left| \mathcal{F}_{z \to \xi} \left(\frac{\varepsilon^2}{4} \Delta u_\varepsilon \left(. + \frac{\varepsilon}{2}\, z \right) \overline{u_\varepsilon \left(. - \frac{\varepsilon}{2}\, z \right)} \exp\left(-\varepsilon \frac{|z|^2}{4} \right) \right) \right| d\xi \right) * \gamma_\varepsilon(x) \\
&\qquad = \left(\int_{\mathbb{R}^d} \left| \mathcal{F}_{z \to \xi} \left(\frac{\varepsilon^2}{4} \Delta u_\varepsilon \left(. + \frac{\varepsilon}{2}\, z \right) \overline{u_\varepsilon \left(. - \frac{\varepsilon}{2}\, z \right)} \right) *_\xi \gamma_\varepsilon \right| d\eta \right) * \gamma_\varepsilon(x) \\
&\qquad \le \left(\int_{\mathbb{R}^d} \left| \mathcal{F}_{z \to \xi} \left(\frac{\varepsilon^2}{4} \Delta u_\varepsilon \left(. + \frac{\varepsilon}{2}\, z \right) \overline{u_\varepsilon \left(. - \frac{\varepsilon}{2}\, z \right)} \right) \right| *_\xi \gamma_\varepsilon \, d\eta \right) * \gamma_\varepsilon(x) \\
&\qquad \le \left(\left|\left| \mathcal{F}_{z \to \xi} \left(\frac{\varepsilon^2}{4} \Delta u_\varepsilon \left(. + \frac{\varepsilon}{2}\, z \right) \overline{u_\varepsilon \left(. - \frac{\varepsilon}{2}\, z \right)} \right) \right|\right|_{L^\infty_\xi} ||\gamma_\varepsilon||_{L^1} \right) * \gamma_\varepsilon(x) \\
&\qquad \le C \left(\left|\left| \frac{\varepsilon^2}{4} \Delta u_\varepsilon \left(. + \frac{\varepsilon}{2}\, z \right) \overline{u_\varepsilon \left(. - \frac{\varepsilon}{2}\, z \right)} \right|\right|_{L^1_z} ||\gamma_\varepsilon||_{L^1} \right) * \gamma_\varepsilon(x) \\
&\qquad \le C \left|\left| \frac{\varepsilon^2}{4} \Delta u_\varepsilon \left(x + \frac{\varepsilon}{2}\, z \right) \overline{u_\varepsilon \left(x - \frac{\varepsilon}{2}\, z \right)} \right|\right|_{L^\infty_x L^1_z} ||\gamma_\varepsilon||_{L^1}\, ||\gamma_\varepsilon||_{L^1} < \infty.
\end{aligned}
$$

Then, we can come back to our equality.

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |\xi|^2\, W_\varepsilon^H(x,\xi)\, d\xi = \left(\int_{\mathbb{R}^d} |\xi|^2\, W_\varepsilon *_\xi \gamma_\varepsilon \, d\xi \right) *_x \gamma_\varepsilon(x) \\
&\qquad = - \left(\int_{\mathbb{R}^d} \mathcal{F}_{z \to \xi} \left(\Delta_z \left(u_\varepsilon \left(x + \frac{\varepsilon}{2}\, z \right) \overline{u_\varepsilon \left(x - \frac{\varepsilon}{2}\, z \right)} \exp\left(-\varepsilon \frac{|z|^2}{4} \right) \right) \right) d\xi \right) *_x \gamma_\varepsilon(x) \\
&\qquad = - \left[\Delta_z \left(u_\varepsilon \left(x + \frac{\varepsilon}{2}\, z \right) \overline{u_\varepsilon \left(x - \frac{\varepsilon}{2}\, z \right)} \exp\left(-\varepsilon \frac{|z|^2}{4} \right) \right) \right] \Big|_{z=0} *_x \gamma_\varepsilon(x)
\end{aligned}
$$

Using the previous calculus of $\Delta_z \left(u_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)\overline{u_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)}\exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right)$, we obtain

$$\begin{aligned}\Delta_z\left(v_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{v_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}\exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right)\Bigg|_{z=0} &= \frac{\varepsilon^2}{4}\Big[\Delta u_\varepsilon(x)\,\overline{u_\varepsilon(x)}+u_\varepsilon(x)\,\overline{\Delta u_\varepsilon(x)}\\ &\qquad -2|\nabla u_\varepsilon(x)|^2\Big]-\frac{\varepsilon d}{2}|u_\varepsilon(x)|^2\\ &= \frac{\varepsilon^2}{4}\left[\Delta\left(|u_\varepsilon|^2\right)(x)-4|\nabla u_\varepsilon(x)|^2\right]\\ &\qquad -\frac{\varepsilon d}{2}|u_\varepsilon(x)|^2.\end{aligned}$$

Therefore, knowing that $\gamma_\varepsilon\in\mathcal{S}(\mathbb{R}^d)$, we can pass the Δ to the other side of the convolution and get

$$\int_{\mathbb{R}^d}|\xi|^2\,W_\varepsilon^H(x,\xi)\,d\xi=\varepsilon^2\,|\nabla u_\varepsilon|^2*\gamma_\varepsilon(x)-\frac{\varepsilon^2}{4}\,|u_\varepsilon|^2*\Delta\gamma_\varepsilon(x)+\frac{\varepsilon d}{2}|u_\varepsilon|^2*\gamma_\varepsilon(x)$$

which yields to the first equality. Keeping in mind that $\gamma_\varepsilon\in\mathcal{S}$, integrating in x yields

$$\begin{aligned}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\xi|^2\,W_\varepsilon^H(x,\xi)\,d\xi dx &= \varepsilon^2\int_{\mathbb{R}^d}|\nabla u_\varepsilon(x)|^2\,dx\int_{\mathbb{R}^d}\gamma_\varepsilon(y)\,dy-\frac{\varepsilon^2}{4}\int_{\mathbb{R}^d}|u_\varepsilon(x)|^2\,dx\int_{\mathbb{R}^d}\Delta\gamma_\varepsilon(y)\,dy\\ &\qquad +\frac{\varepsilon d}{2}\int_{\mathbb{R}^d}|u_\varepsilon(x)|^2\,dx\int_{\mathbb{R}^d}\gamma_\varepsilon(y)\,dy\\ &= \varepsilon^2||\nabla u_\varepsilon||_{L^2}^2+\frac{\varepsilon d}{2}||u_\varepsilon||_{L^2}^2.\end{aligned}$$

A.1.2 Scalar second momentum: H^1 case

In the same way, we can still consider $\int_{\mathbb{R}^d}|\xi|^2\,W_\varepsilon^H(x,\xi)\,d\xi$ even for $u_\varepsilon\in H^1$. However, it could still be equal to $+\infty$. The first part will be to show that this is not the case.

For fixed $\varepsilon>0$, take a sequence of functions $u_{\varepsilon,k}$ in H^2 converging to u_ε in H^1 when $k\to\infty$. Using the notation $W_{\varepsilon,k}$ (resp. $W_{\varepsilon,k}^H$) for the Wigner Transform (resp. the Husimi Transform) of the functions of the sequence, we first show that they converge uniformly to the Wigner Transform W_ε (resp. the Husimi Transform W_ε^H) of u_ε. For $x,\xi\in\mathbb{R}^d$

$$\begin{aligned}|W_{\varepsilon,k}(x,\xi)-W_\varepsilon(x,\xi)| &= |\mathcal{F}_z\tilde\rho_{\varepsilon,k}(x,\xi)-\mathcal{F}_z\tilde\rho_\varepsilon(x,\xi)|\\ &\le C||\tilde\rho_{\varepsilon,k}(x,.)-\tilde\rho_\varepsilon(x,.)||_{L^1}\\ &\le C\left|\left|u_{\varepsilon,k}(x+\frac{\varepsilon z}{2})\,\overline{u_{\varepsilon,k}(x-\frac{\varepsilon z}{2})}-u_\varepsilon(x+\frac{\varepsilon z}{2})\,\overline{u_\varepsilon(x-\frac{\varepsilon z}{2})}\right|\right|_{L^1_z}\\ &\le C\left(\left|\left|u_{\varepsilon,k}(x+\frac{\varepsilon z}{2})\,\left(\overline{u_{\varepsilon,k}(x-\frac{\varepsilon z}{2})-u_\varepsilon(x-\frac{\varepsilon z}{2})}\right)\right|\right|_{L^1_z}\right.\\ &\qquad\left.+\left|\left|\left(u_{\varepsilon,k}(x+\frac{\varepsilon z}{2})-u_\varepsilon(x+\frac{\varepsilon z}{2})\right)\overline{u_\varepsilon(x-\frac{\varepsilon z}{2})}\right|\right|_{L^1_z}\right)\\ &\le C\left(\left|\left|u_{\varepsilon,k}(x+\frac{\varepsilon z}{2})\right|\right|_{L^2_z}\left|\left|u_{\varepsilon,k}(x-\frac{\varepsilon z}{2})-u_\varepsilon(x-\frac{\varepsilon z}{2})\right|\right|_{L^2_z}\right.\\ &\qquad\left.+\left|\left|u_{\varepsilon,k}(x+\frac{\varepsilon z}{2})-u_\varepsilon(x+\frac{\varepsilon z}{2})\right|\right|_{L^2_z}\left|\left|u_\varepsilon(x-\frac{\varepsilon z}{2})\right|\right|_{L^2_z}\right)\\ &\le C_\varepsilon\left(||u_{\varepsilon,k}||_{L^2}\,||u_{\varepsilon,k}-u_\varepsilon||_{L^2}+||u_{\varepsilon,k}-u_\varepsilon||_{L^2}\,||u_\varepsilon||_{L^2}\right)\\ &\le C_\varepsilon\,||u_{\varepsilon,k}-u_\varepsilon||_{L^2}\,.\end{aligned}$$

Therefore, $W_{\varepsilon,k}$ converges uniformly to W_ε with the bound

$$||W_{\varepsilon,k} - W_\varepsilon||_{L^\infty} \leq C_\varepsilon \, ||u_{\varepsilon,k} - u_\varepsilon||_{L^2} \, ,$$

and the same kind of bound hold for the Husimi Transform:

$$||W^H_{\varepsilon,k} - W^H_\varepsilon||_{L^\infty} = ||\, (W_{\varepsilon,k} - W_\varepsilon) * G_\varepsilon||_{L^\infty} \leq ||W_{\varepsilon,k} - W_\varepsilon||_{L^\infty} ||G_\varepsilon||_{L^1} \leq C_\varepsilon \, ||u_{\varepsilon,k} - u_\varepsilon||_{L^2} \, .$$

Thus, we can use Fatou lemma for $|\xi|^2 W^H_{\varepsilon,k}(x,\xi)$, which leads to

$$\int_{\mathbb{R}^d} |\xi|^2 \, W^H_\varepsilon(x,\xi) \, d\xi \leq \liminf_{k\to\infty} \int_{\mathbb{R}^d} |\xi|^2 \, W^H_{\varepsilon,k}(x,\xi) \, d\xi.$$

The previous calculus yields

$$\int_{\mathbb{R}^d} |\xi|^2 \, W^H_{\varepsilon,k}(x,\xi) \, d\xi = \varepsilon^2 \, |\nabla u_{\varepsilon,k}|^2 * \gamma_\varepsilon(x) - \frac{\varepsilon^2}{4} \, |u_{\varepsilon,k}|^2 * \Delta\gamma_\varepsilon(x) + \frac{\varepsilon d}{2} |u_{\varepsilon,k}|^2 * \gamma_\varepsilon(x).$$

But $u_{\varepsilon,k} \underset{k\to\infty}{\longrightarrow} u_\varepsilon$ in H^1, so $|\nabla u_{\varepsilon,k}|^2 \underset{k\to\infty}{\longrightarrow} |\nabla u_\varepsilon|^2$ and $|u_{\varepsilon,k}|^2 \underset{k\to\infty}{\longrightarrow} |u_\varepsilon|^2$ in L^1. Those limits show that the right hand side of the previous equality goes to the good term when $k \to \infty$, which means that

$$\int_{\mathbb{R}^d} |\xi|^2 \, W^H_\varepsilon(x,\xi) \, d\xi \leq \varepsilon^2 \, |\nabla u_\varepsilon|^2 * \gamma_\varepsilon(x) - \frac{\varepsilon^2}{4} \, |u_\varepsilon|^2 * \Delta\gamma_\varepsilon(x) + \frac{\varepsilon d}{2} |u_\varepsilon|^2 * \gamma_\varepsilon(x) < \infty.$$

Therefore, the map

$$\begin{aligned} H^1 &\to \mathbb{R}^+ \\ u_\varepsilon &\mapsto \int_{\mathbb{R}^d} |\xi|^2 \, W^H_\varepsilon(x,\xi) \, d\xi \end{aligned}$$

is well-defined for every $x \in \mathbb{R}^d$. Moreover, it is a non-negative quadratic form because W_ε and then also W^H_ε are quadratic. Furthermore, it is continuous thanks to the previous inequality which leads to

$$\int_{\mathbb{R}^d} |\xi|^2 \, W^H_\varepsilon(x,\xi) \, d\xi \leq C_\varepsilon ||u_\varepsilon||^2_{H^1}.$$

Thus, the equality (2.19), which is true in H^2 dense subspace in H^1, also holds in H^1 since the right hand side is also continuous.

A.1.3 Vector second momentum: H^2 case

With the same hypothesis, we can consider $\int_{\mathbb{R}^d} \xi_i \xi_j \, W^H_\varepsilon(x,\xi) \, d\xi$ as we now know that $\xi_i \xi_j \, W^H_\varepsilon(x,\xi)$ is integrable thanks to the previous equality, and in the same way, we have for $u_\varepsilon \in H^2$ and for every $x \in \mathbb{R}^d$:

$$\begin{aligned} \int_{\mathbb{R}^d} \xi_i \xi_j \, W^H_\varepsilon(x,\xi) \, d\xi &= \int_{\mathbb{R}^d} \xi_i \xi_j \, (W_\varepsilon *_\xi \gamma_\varepsilon *_x \gamma_\varepsilon)(x,\xi) \, d\xi \\ &= \left(\int_{\mathbb{R}^d} \xi_i \xi_j \, W_\varepsilon *_\xi \gamma_\varepsilon \, d\xi \right) *_x \gamma_\varepsilon(x), \end{aligned}$$

the exchange of integral is rigorous with the same kind of estimation as previously. Moreover,

$$\begin{aligned} \int_{\mathbb{R}^d} \xi_i \xi_j \, W_\varepsilon(x,.) *_\xi \gamma_\varepsilon \, d\xi &= -\int_{\mathbb{R}^d} \mathcal{F}_{z\to\xi} \left(\partial_{z_i} \partial_{z_j} \left(u_\varepsilon \left(x + \frac{\varepsilon}{2} \, z \right) \overline{u_\varepsilon \left(x - \frac{\varepsilon}{2} \, z \right)} \, \exp\left(-\varepsilon \frac{|z|^2}{4} \right) \right) \right) \, d\xi \\ &= - \left[\partial_{z_i} \partial_{z_j} \left(u_\varepsilon \left(x + \frac{\varepsilon}{2} \, z \right) \overline{u_\varepsilon \left(x - \frac{\varepsilon}{2} \, z \right)} \, \exp\left(-\varepsilon \frac{|z|^2}{4} \right) \right) \right] \Bigg|_{z=0} . \end{aligned}$$

Then we have

$$\partial_{z_i}\partial_{z_j}\left(u_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{u_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}\exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right)$$
$$=\frac{\varepsilon}{2}\partial_{z_i}\left(\partial_j u_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{u_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}\exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right.$$
$$\left.-u_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{\partial_j u_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}\exp\left(-\varepsilon\frac{|z|^2}{4}\right)-u_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{u_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}z_j\exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right)$$

$$=\frac{\varepsilon^2}{4}\left[\partial_i\partial_j u_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{u_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}\exp\left(-\varepsilon\frac{|z|^2}{4}\right)+u_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{\partial_i\partial_j u_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}\exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right.$$
$$+u_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{u_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}z_iz_j\exp\left(-\varepsilon\frac{|z|^2}{4}\right)-\partial_i u_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{\partial_j u_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}\exp\left(-\varepsilon\frac{|z|^2}{4}\right)$$
$$-\partial_j u_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{\partial_i u_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}\exp\left(-\varepsilon\frac{|z|^2}{4}\right)-\partial_i u_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{u_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}z_j\exp\left(-\varepsilon\frac{|z|^2}{4}\right)$$
$$-\partial_j u_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{u_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}z_i\exp\left(-\varepsilon\frac{|z|^2}{4}\right)+u_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{\partial_i u_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}z_j\exp\left(-\varepsilon\frac{|z|^2}{4}\right)$$
$$\left.+u_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{\partial_j u_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}z_i\exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right]-\frac{\varepsilon\delta_{ij}}{2}u_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{u_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}\exp\left(-\varepsilon\frac{|z|^2}{4}\right),$$

which yields

$$\left[\partial_{z_i}\partial_{z_j}\left(u_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{u_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}\exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right)\right]\Big|_{z=0}=\frac{\varepsilon^2}{4}\left[\partial_i\partial_j u_\varepsilon(x)\,\overline{u_\varepsilon(x)}+u_\varepsilon(x)\,\overline{\partial_i\partial_j u_\varepsilon(x)}\right.$$
$$\left.-\partial_i u_\varepsilon(x)\,\overline{\partial_j u_\varepsilon(x)}-\partial_j u_\varepsilon(x)\,\overline{\partial_i u_\varepsilon(x)}\right]$$
$$-\frac{\varepsilon\delta_{ij}}{2}u_\varepsilon(x)\,\overline{u_\varepsilon(x)}$$
$$=\frac{\varepsilon^2}{4}\left[\partial_i\partial_j\left(|u_\varepsilon|^2\right)(x)-4\,\mathrm{Re}\left(\partial_i u_\varepsilon(x)\,\overline{\partial_j u_\varepsilon(x)}\right)\right]$$
$$-\frac{\varepsilon\delta_{ij}}{2}|u_\varepsilon(x)|^2.$$

Therefore, in the same way as previously,

$$\int_{\mathbb{R}^d}\xi_i\xi_j\,W_\varepsilon^H(x,\xi)\,d\xi=\varepsilon^2\,\mathrm{Re}\left(\partial_i u_\varepsilon(x)\,\overline{\partial_j u_\varepsilon(x)}\right)*\gamma_\varepsilon(x)-\frac{\varepsilon^2}{4}|u_\varepsilon|^2*\partial_i\partial_j\gamma_\varepsilon(x)+\frac{\varepsilon\delta_{ij}}{2}|u_\varepsilon|^2*\gamma_\varepsilon(x),$$

and

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\xi_i\xi_j\,W_\varepsilon^H(x,\xi)\,d\xi dx=\varepsilon^2\int_{\mathbb{R}^d}\mathrm{Re}\left(\partial_i u_\varepsilon(x)\,\overline{\partial_j u_\varepsilon(x)}\right)\,dx+\frac{\varepsilon\delta_{ij}}{2}||u_\varepsilon||_{L^2}^2.$$

A.1.4 Vector second momentum: H^1 case

The generalization of this equality is similar to the end of the previous generalization for the scalar second momentum. The map

$$H^1\to\mathbb{R}$$
$$u_\varepsilon\mapsto\int_{\mathbb{R}^d}\xi_i\xi_j\,W_\varepsilon^H(x,\xi)\,d\xi$$

is a well-defined, continuous quadratic form thanks to the previous equality for the scalar second momentum. Then, the equality made for $u_\varepsilon\in H^2$ also hold for $u_\varepsilon\in H^1$.

A.2 Second part: first momentum in ξ

We know that $\int_{\mathbb{R}^d} |\xi|^2 \, W_\varepsilon^H(x,\xi)\, d\xi < \infty$ by the previous proof and also that $\int_{\mathbb{R}^d} W_\varepsilon^H(x,\xi)\, d\xi < \infty$, therefore we can consider $\int_{\mathbb{R}^d} \xi \, W_\varepsilon^H(x,\xi)\, d\xi$. Then:

$$\begin{aligned}\int_{\mathbb{R}^d} \xi \, W_\varepsilon^H(x,\xi)\, d\xi &= \int_{\mathbb{R}^d} \xi \, (W_\varepsilon *_\xi \gamma_\varepsilon *_x \gamma_\varepsilon)(x,\xi)\, d\xi \\ &= \left(\int_{\mathbb{R}^d} \xi \, W_\varepsilon *_\xi \gamma_\varepsilon \, d\xi \right) *_x \gamma_\varepsilon,\end{aligned}$$

the integral exchange being rigorous with the same kind of calculus as before, which infers that:

$$\begin{aligned}\int_{\mathbb{R}^d} \xi \, W_\varepsilon^H(x,\xi)\, d\xi &= \left(-i \, \nabla_z \left(u_\varepsilon \left(. + \frac{\varepsilon}{2} z \right) \overline{u_\varepsilon \left(. - \frac{\varepsilon}{2} z \right)} \exp\left(-\varepsilon \frac{|z|^2}{4} \right) \right) \Big|_{z=0} \right) *_x \gamma_\varepsilon \\ &= \varepsilon \, \mathrm{Im} \left(\nabla u_\varepsilon \, \overline{u_\varepsilon} \right) * \gamma_\varepsilon(x)\end{aligned}$$

and therefore the first equality, the second one being obvious by integrating this result.

A.3 Third part: second momentum in x

In the same way, as W_ε^H is non-negative, we have, thanks to Proposition 2.2,

$$\begin{aligned}\iint_{\mathbb{R}^d \times \mathbb{R}^d} |x|^2 \, W_\varepsilon^H(x,\xi)\, dx d\xi &= \int_{\mathbb{R}^d} |x|^2 \left(\int_{\mathbb{R}^d} W_\varepsilon^H(x,\xi)\, d\xi \right) dx \\ &= \int_{\mathbb{R}^d} |x|^2 \, |u_\varepsilon|^2 * \gamma_\varepsilon(x)\, dx \\ &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x|^2 \, |u_\varepsilon(x-y)|^2 * \gamma_\varepsilon(y)\, dy dx.\end{aligned}$$

Therefore

$$\begin{aligned}&\iint_{\mathbb{R}^d \times \mathbb{R}^d} |x|^2 \, W_\varepsilon^H(x,\xi)\, dx d\xi - \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 \, |u_\varepsilon(x-y)|^2 * \gamma_\varepsilon(y)\, dy dx \\ &\quad = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (2\,(x-y)\cdot y + |y|^2) \, |u_\varepsilon(x-y)|^2 * \gamma_\varepsilon(y)\, dy dx \\ &\quad = 2 \left(\int_{\mathbb{R}^d} x \, |u_\varepsilon(x)|^2 \, dx \right) \cdot \left(\int_{\mathbb{R}^d} y \, \gamma_\varepsilon(y)\, dy \right) + ||u_\varepsilon||_{L^2}^2 \int_{\mathbb{R}^d} |y|^2 \, \gamma_\varepsilon(y)\, dy \\ &\quad = \frac{\varepsilon d}{2} \, ||u_\varepsilon||_{L^2}^2\end{aligned}$$

because $\int_{\mathbb{R}^d} y \, \gamma_\varepsilon(y)\, dy = 0$ and $\int_{\mathbb{R}^d} |y|^2 \, \gamma_\varepsilon(y)\, dy = \frac{\varepsilon d}{2}$. The proof is complete as soon as we observe that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 \, |u_\varepsilon(x-y)|^2 * \gamma_\varepsilon(y)\, dy dx = |||x|^2 |u_\varepsilon(x)|^2||_{L^1}$$

Appendix B

Proof of the convergence of the p-th moment

We now prove the convergence of the p-th moment of $\tilde{\rho}(t)$ to the p-th moment of γ^2 for all $p \in [1,2)$. The proof is actually a straightforward consequence of the following lemma:

Lemma B.1 (Convergence of the p-th moment)**.** *Let (f_n) a sequence in $L^1(\mathbb{R}^d)$ and $f \in L^1(\mathbb{R}^d)$ such that $f_n \underset{n\to\infty}{\rightharpoonup} f$ in L^1. Assume also that there exists $q > 0$ and C independent of n such that $\int_{\mathbb{R}^d} |x|^q \, |f_n(x)| \, dx \leq C$. Then, for all $p \in (0,q)$,*

$$\int_{\mathbb{R}^d} |x|^p \, f_n(x) \, dx \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} |x|^p \, f(x) \, dx$$

Proof. Take $\varepsilon > 0$, $R > 0$ and $p \in (0,q)$. Then cutting each integrals in two:

$$\left| \int_{\mathbb{R}^d} |x|^p \, f_n(x) \, dx - \int_{\mathbb{R}^d} |x|^p \, f(x) \, dx \right| \leq \left| \int_{B(0,R)} |x|^p \, f_n(x) \, dx - \int_{B(0,R)} |x|^p \, f(x) \, dx \right| + \left| \int_{B(0,R)^c} |x|^p \, f_n(x) \, dx - \int_{B(0,R)^c} |x|^p \, f(x) \, dx \right| .$$

But then,

$$\begin{aligned} \left| \int_{B(0,R)^c} |x|^p \, f_n(x) \, dx \right| &\leq \int_{B(0,R)^c} |x|^p \, |f_n|(x) \, dx \\ &\leq \int_{B(0,R)^c} |x|^p \frac{|x|^{q-p}}{R^{q-p}} \, |f_n(x)| \, dx \\ &\leq \frac{1}{R^{q-p}} \int_{\mathbb{R}^d} |x|^q \, |f_n(x)| \, dx \\ &\leq \frac{C}{R^{q-p}} \end{aligned}$$

Thanks to the bound $\int_{\mathbb{R}^d} |x|^q \, |f_n(x)| \, dx \leq C$, it is well known that $\int_{\mathbb{R}^d} |x|^q \, |f(x)| \, dx \leq C$, and therefore we have the same bound as previously for $\int_{B(0,R)^c} |x|^p \, f(x) \, dx$. Then, take $R > 0$ such that $\frac{2C}{R^{q-p}} \leq \frac{\varepsilon}{2}$. Therefore,

$$\left| \int_{\mathbb{R}^d} |x|^p \, f_n(x) \, dx - \int_{\mathbb{R}^d} |x|^p \, f(x) \, dx \right| \leq \left| \int_{B(0,R)} |x|^p \, f_n(x) \, dx - \int_{B(0,R)} |x|^p \, f(x) \, dx \right| + \frac{\varepsilon}{2}$$

Finally,

$$\int_{B(0,R)} |x|^p \, f_n(x) \, dx = \int_{\mathbb{R}^d} \mathbb{1}_{B(0,R)} |x|^p \, f_n(x) \, dx \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \mathbb{1}_{B(0,R)} |x|^p \, f(x) \, dx = \int_{B(0,R)} |x|^p \, f(x) \, dx,$$

because $\mathbb{1}_{B(0,R)} |x|^p \in L^\infty(\mathbb{R}^d)$. Therefore, take N such that for all $n \geq N$,

$$\left| \int_{B(0,R)} |x|^p \, f_n(x) \, dx - \int_{B(0,R)} |x|^p \, f(x) \, dx \right| \leq \frac{\varepsilon}{2},$$

which yields for all $n \geq N$,

$$\left| \int_{\mathbb{R}^d} |x|^p \, f_n(x) \, dx - \int_{\mathbb{R}^d} |x|^p \, f(x) \, dx \right| \leq \varepsilon.$$

The convergence is proven. □

Appendix C

Proof of Proposition 2.8

The main part of this proof is to prove the second part of the Proposition. Indeed, the computations that will be done can be done reversely, which will show the first part. The only thing that remains to prove is that c_1 solution to (2.34) is $C^\infty(\mathbb{R}^+)$, but this has already been done in [7].

Therefore, with the notation and hypothesis of the second part of Proposition 2.8, we calculate:

$$\partial_t f(t,x,\xi) = \left[-d\frac{\dot{c}_1(t)}{c_1(t)} - d\frac{\partial_t c_2(t,x)}{c_2(t,x)} + 2\,\frac{\dot{c}_1(t)\,|x-b_1(t)|^2}{c_1(t)^3} + 2\,\frac{\dot{b}_1(t)\cdot(x-b_1(t))}{c_1(t)^2} \right.$$
$$\left. +2\,\frac{\partial_t c_2(t,x)\,|\xi-b_2(t,x)|^2}{c_2(t,x)^3} + 2\,\frac{\partial_t b_2(t,x)\cdot(\xi-b_2(t,x))}{c_2(t,x)^2} \right] f(t,x,\xi),$$

$$\partial_x f(t,x,\xi) = \left[-2\,\frac{x-b_1(t)}{c_1(t)^2} + 2\,\partial_x b_2(t,x)\frac{\xi-b_2(t,x)}{c_2(t,x)^2} + 2\,\partial_x c_2(t,x)\frac{(\xi-b_2(t,x))^2}{c_2(t,x)^3} \right] f(t,x,\xi),$$

$$\partial_\xi f(t,x,\xi) = -2\,\frac{\xi-b_2(t,x)}{c_2(t)^2} f(t,x,\xi),$$

$$\rho(t,x) = \frac{1}{\sqrt{\pi}\,c_1(t)} e^{-\frac{(x-b_1(t))^2}{c_1(t)^2}} \Rightarrow \ln\rho(t,x) = -\frac{(x-b_1(t))^2}{c_1(t)^2} - \ln(\sqrt{\pi}\,c_1(t))$$
$$\Rightarrow \partial_x(\ln\rho)(t,x) = -2\,\frac{x-b_1(t)}{c_1(t)}$$

Putting all those equalities in the equation (1.1) leads us to an equation which is of the form

$$P(t,x,\xi)f(t,x,\xi) = 0$$

where P is a function such that for every (t,x), $P(t,x,.)$ is polynomial of degree at most 3. Since $f(t,x,\xi) > 0$ for every (t,x,ξ), there holds $P = 0$ and therefore for every (t,x), the coefficients of the polynomial function $p(t,x,.)$ are null. In particular, the coefficient of higher degree comes from the term $\xi\,\partial_x f$ and is

$$2\,\frac{\partial_x c_2(t,x)}{c_2(t,x)^3}.$$

We supposed that $c_2 > 0$, therefore

$$\partial_x c_2(t,x) = 0, \qquad \text{for all } (t,x),$$

and thus c_2 does not depend on t. We now take a more suitable basis to get null coefficients for the polynomial function, of degree at most 2, $\xi \mapsto P(t,x,\xi)$: $\big((\xi - b_2(t,x))^2, \xi - b_2(t,x), 1\big)$. Again, the coefficients in this basis are null, which yields for $(\xi - b_2(t,x))^2$:

$$2\frac{\dot{c}_2(t)}{c_2(t)^3} + 2\frac{\partial_x b_2(t,x)}{c_2(t,x)^2} = 0.$$

This equation leads to

$$\partial_x b_2(t,x) = -\frac{\dot{c}_2(t)}{c_2(t)},$$

and then, there exists a function $p_0 = p_0(t)$ such that:

$$b_2(t,x) = -\frac{\dot{c}_2(t)}{c_2(t)}\, x + p_0(t), \qquad \text{for all } (t,x).$$

The assumption on the regularity of b_2 shows that $p_0 \in \mathcal{C}^1([0,T))$. But then, we also get thanks to the same assumption and the assumption on the regularity of c_2:

$$\dot{c}_2(t) = c_2(t)\,(p_0(t) - b_2(t,1)) \in \mathcal{C}^1([0,T)).$$

Therefore, $c_2 \in \mathcal{C}^2([0,T))$. Now, watching the coefficient for $(\xi - b_2(t,x))$, we get

$$2\,\frac{\partial_t b_2(t,x)}{c_2(t)^2} - 2\,\frac{x - b_1(t)}{c_1(t)^2} + 2\,\frac{b_2(t,x)\,\partial_x b_2(t,x)}{c_2(t)^2} - 4\lambda\,\frac{x - b_1(t)}{c_1(t)^2\,c_2(t)^2} = 0, \qquad \text{for all } (t,x).$$

In terms of $\partial_t b_2$, this reads

$$\begin{aligned}
\partial_t b_2(t,x) &= \left(1 + \frac{2\lambda}{c_2(t)^2}\right)\frac{c_2(t)^2}{c_1(t)^2}\,(x - b_1(t)) - b_2(t,x)\,\partial_x b_2(t,x)\\
&= \left[\left(1 + \frac{2\lambda}{c_2(t)^2}\right)\frac{c_2(t)^2}{c_1(t)^2} - \frac{\dot{c}_2(t)^2}{c_2(t)^2}\right] x - \left(1 + \frac{2\lambda}{c_2(t)^2}\right)\frac{c_2(t)^2}{c_1(t)^2}\,b_1(t) + \frac{\dot{c}_2(t)}{c_2(t)}p_0(t).
\end{aligned}$$

However, the previous expression of b_2 allows us to compute another expression of $\partial_t b_2$:

$$\partial_t b_2(t,x) = \left(-\frac{\ddot{c}_2(t)}{c_2(t)} + \frac{\dot{c}_2(t)^2}{c_2(t)^2}\right) x + \dot{p}_0(t).$$

This yields the following system of equations for all $t \geq 0$:

$$\begin{cases}
\left(1 + \dfrac{2\lambda}{c_2(t)^2}\right)\dfrac{c_2(t)^2}{c_1(t)^2} = -\dfrac{\ddot{c}_2(t)}{c_2(t)} + 2\,\dfrac{\dot{c}_2(t)^2}{c_2(t)^2}, & \text{(C.1)}\\
\dot{p}_0(t) = -\left(1 + \dfrac{2\lambda}{c_2(t)^2}\right)\dfrac{c_2(t)^2}{c_1(t)^2}\,b_1(t) + \dfrac{\dot{c}_2(t)}{c_2(t)}p_0(t). & \text{(C.2)}
\end{cases}$$

In particular, the second equation shows that $\dot{p}_0 \in \mathcal{C}^1([0,T))$ (since the right-hand side is) and is actually an ordinary differential equation of order 1. The solution is well-known as soon as we remark that $\frac{\dot{c}_2}{c_2} = \frac{\mathrm{d}}{\mathrm{d}t}(\ln c_2)$ and reads:

$$p_0(t) = c_2(t)\left(C_0 - \int_0^t \left(1 + \frac{2\lambda}{c_2(s)^2}\right)\frac{c_2(s)^2}{c_1(s)^2}\,\frac{b_1(s)}{c_2(s)}\,ds\right)$$

and thanks to the first equation, we can expand it:

$$\begin{aligned} p_0(t) &= c_2(t)\left(C_0 - \int_0^t \left(-\frac{\ddot{c}_2(s)}{c_2(s)} + 2\,\frac{\dot{c}_2(s)^2}{c_2(s)^2}\right)\frac{b_1(s)}{c_2(s)}\,ds\right) \\ &= c_2(t)\left(C_0 + \int_0^t \frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{1}{c_2(s)}\right) b_1(s)\,ds\right) \\ &= c_2(t)\,C_1 + \frac{\dot{c}_2(t)}{c_2(t)}\,b_1(t) - c_2(t)\int_0^t \frac{\dot{c}_2(s)}{c_2(s)^2}\,\dot{b}_1(s)\,ds, \end{aligned}$$

with $C_1 = C_0 - \frac{\dot{c}_2(0)}{c_2(0)^2}\,b_1(0)$ with an integration by parts. Last, the constant in ξ gives the following equation:

$$-\frac{\dot{c}_1(t)}{c_1(t)} - \frac{\dot{c}_2(t)}{c_2(t)} + 2\,\frac{\dot{c}_1(t)}{c_1(t)^3}\,(x - b_1(t))^2 + 2\frac{\dot{b}_1(t)}{c_1(t)^2}\,(x - b_1(t)) - 2\,\frac{b_2(t,x)}{c_1(t)^2}(x - b_1(t)) = 0 \quad \text{(C.3)}$$

But since we know that b_2 is affine in x, the left-hand side is a polynomial function in x of degree 2 for all $t \in [0, T)$. Therefore, the coefficients in a suitable basis are null. This time, we take the basis: $\big((x - b_1(t))^2, x - b_1(t), 1\big)$. For the first one and for the constants, we get

$$\begin{aligned} 2\,\frac{\dot{c}_1(t)}{c_1(t)^3} + 2\,\frac{\dot{c}_2(t)}{c_2(t)^3} &= 0, \\ -\frac{\dot{c}_1(t)}{c_1(t)} - \frac{\dot{c}_2(t)}{c_2(t)} &= 0. \end{aligned}$$

Those 2 equations actually reduce in a single one, for instance

$$\frac{\mathrm{d}}{\mathrm{d}t}(c_1\,c_2) = 0,$$

and therefore, for all $t \in [0, T)$,

$$c_1(t)\,c_2(t) = c_1(0)\,c_2(0) =: \tilde{C} > 0.$$

We already know that c_2 is $\mathcal{C}^2$ and positive, therefore so is c_1. Coming back to (C.1), we now have

$$\ddot{c}_2 = 2\,\frac{\dot{c}_2^2}{c_2} - \frac{2\lambda}{\tilde{C}}c_2^3 - \frac{c_2^5}{\tilde{C}},$$

which reads in terms of c_1

$$\ddot{c}_1 = \frac{2\lambda}{c_1} + \frac{\tilde{C}^2}{c_1^3},$$

which is (2.34). Last, the final equation we have comes from the coefficient for $(x - b_1(t))$:

$$2\,\frac{\dot{b}_1(t)}{c_1(t)^2} + 2\,\frac{\dot{c}_2(t)}{c_1(t)^2\,c_2(t)}\,b_1(t) - \frac{p_0(t)}{c_1(t)^2} = 0, \qquad \text{for all } t \geq 0.$$

This leads to

$$\dot{b}_1 = -\frac{\dot{c}_2}{c_2}\,b_1 + p_0.$$

All the terms in the right-hand side are $\mathcal{C}^1([0, T))$, therefore so is $\dot{b}_1$, which yields to the $\mathcal{C}^2$-regularity of b_1. Therefore, we can again expand the expression for p_0 found previously with another integration by parts:

$$p_0(t) = C_2\,c_2(t) + \frac{\dot{c}_2(t)}{c_2(t)}\,b_1(t) + \dot{b}_1(t) - c_2(t)\int_0^t \frac{\ddot{b}_2(s)}{c_2(s)}\,ds,$$

with $C_2 = C_1 + \frac{1}{c_2(0)}$ Plugging this expression of p_0 into the expression of $\dot{b}_1$ leads to

$$C_2\, c_2(t) = c_2(t) \int_0^t \frac{\ddot{b}_2(s)}{c_2(s)}\, ds.$$

Since $c_2 > 0$, we then obtain $C_2 = 0$ and $\frac{\ddot{b}_2}{c_2} = 0$, which is $\ddot{b}_2 = 0$. Thus, there exists B_0, B_1 constants such that

$$b_1 = B_1\, t + B_0,$$

and this gives the final expression for p_0 (and therefore for b_2):

$$p_0(t) = (B_1\, t + B_0)\frac{\dot{c}_2(t)}{c_2(t)} + B_1.$$

□

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