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Equation de Schrödinger non-linéaire avec non-linéarité logarithmique

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Résumé. Cette thèse est centrée sur l'analyse de l'équation de Schrödinger non-linéaire avec non-linéarité logarithmique (logNLS). Dans un premier temps, nous étudions le comportement en temps long en régime défocalisant, dont un comportement universel a été observé. Nous précisons ce comportement par une vitesse de convergence optimale en distance de Wasserstein (aussi appelée distance de Kantorovich-Rubinstein). Nous montrons en parallèle que les propriétés de cette équation permettent d'obtenir via la transformée de Wigner un objet limite à la limite semi-classique vérifiant également le même comportement en temps long. Cette commutation entre limite semi-classique et comportement en temps long est une caractéristique inhabituelle pour une équation de Schrödinger. Par la suite, nous nous intéressons au régime focalisant, et plus particulièrement aux intéractions entre solitons (qui sont dans ce cadre des fonctions gaussiennes appelées *Gaussons*) et même plus généralement entre les solutions gaussiennes explicites que cette équation admet. Nous montrons d'abord qu'une solution de logNLS ayant pour donnée initiale une somme de gaussiennes éloignées entre elles reste proche de la somme des solutions gaussiennes correspondantes jusqu'à un temps de l'ordre du carré de la distance minimale entre les gaussiennes. Ensuite, nous démontrons l'existence de multi-Gaussons et même de multi-gaussiennes (solution se comportant en temps grand comme une somme de solutions gaussiennes) ayant une vitesse de convergence plus rapide qu'exponentielle, ainsi que leur unicité sous cette hypothèse de vitesse de convergence. Pour finir, nous effectuons une analyse BKW de cette équation. Les équations limites de cette analyse étant le système d'Euler isotherme, dont la résolution a été faite via les variables de Riemann, nous établissons une théorie de Cauchy en utilisant des inconnues similaires aux variables de Riemann, correspondant à des solutions de la forme $e^{\psi^\varepsilon + \frac{\phi^\varepsilon}{\varepsilon}}$ pour logNLS dans un cadre semi-classique, sous hypothèse d'analycité. Nous montrons en outre que ces variables convergent à la limite semi-classique, et que les fonctions limites sont solutions du système d'Euler isotherme sous la forme "variables de Riemann".

Mots-clés : équations de Schrödinger non-linéaires ; équations dispersives ; non-linéarité logarithmique ; comportement en temps long ; multi-solitons ; techniques d'énergie ; dispersion ; équation de Fokker-Planck ; semi-groupe de Fokker-Planck ; limite semi-classique ; Transformée de Wigner ; analyse BKW ; système d'Euler isotherme ; équations cinétiques ; système d'Euler isotherme cinétique.

Nonlinear Schrödinger equation with logarithmic nonlinearity

Abstract. The topic of this thesis is the analysis of the nonlinear Schrödinger equation with logarithmic nonlinearity (logNLS). Firstly, we study the long-time behavior in defocusing regime. A universal behavior has already been observed in this context. We specify this behavior by an optimal convergence rate in Wasserstein distance (also called Kantorovich-Rubinstein). In parallel, we show that the properties of this equation allows us to obtain, via the Wigner Transform, a limit object in the semiclassical limit satisfying the the same long-time behavior. This commutation between semiclassical limit and long-time behavior is an unusual feature for a Schrödinger equation. Then, we are interested in the focusing regime, and specifically in the interactions between solitons (which are in this case Gaussian functions called *Gaussons*), and more generally between the explicit gaussian solutions that this equation admits. We first show that a solution to logNLS with a sum of gaussian functions which are far away from each other for initial data stays close to the sum of corresponding gaussian solutions until at least a time of order the square of the minimal distance between the Gaussian. Then, we prove the existence of multi-Gaussons and multi-gaussian (solution which behaves in large time like a sum of several gaussian solutions) with a convergence rate faster than exponential, and their uniqueness for such a convergence rate. Last, a WKB analysis is performed for this equation. The limit equations of this analysis are the isotherm Euler system, whose solution have been done thanks to the Riemann variables. We establish a new Cauchy theory by using new variables similar as the Riemann variables, corresponding to solutions of logNLS of the form $e^{\psi^\varepsilon + \frac{\phi^\varepsilon}{\varepsilon}}$ in a semiclassical regime, under an assumption of analycity. We also show that these variables converge in the semiclassical limit, and that the limit functions are solutions to the isothermal Euler system under its "Riemann variables" form.

Key words : nonlinear Schrödinger equations ; dispersive equations ; logarithmic nonlinearity ; long-time behavior ; multi-solitons ; energy techniques ; dispersion ; Fokker-Planck equation ; Fokker-Planck semigroup ; semiclassical analysis ; Wigner Transform ; WKB analysis ; isothermal Euler system ; kinetic equations ; kinetic isothermal Euler system.

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Introduction

Cette introduction commence par un bref rappel du contexte historique de l'équation de Schrödinger, linéaire et non-linéaire, en physique depuis le début du XXe siècle dans la Section 1. Par la suite, on s'intéresse dans la Section 2 à plusieurs aspects mathématiques de l'équation de Schrödinger non-linéaire, pour des non-linéarités de type puissance ou au moins lipschitziennes. Sont essentiellement abordés problème de Cauchy, comportement en temps long et analyse semi-classique pour cette équation, sur lesquels de très nombreux résultats sont connus et rappelés (de manière non-exhaustive) dans cette section. Plusieurs résultats sur ces sujets ont également déjà été prouvés pour l'équation au cœur du sujet de cette thèse, l'équation de Schrödinger logarithmique, et sont explicités dans la Section 3. De nombreuses problématiques restent cependant ouvertes sur cette équation : nous présentons dans la Section 4 celles qui nous ont intéressé durant cette thèse. La Section 5 est consacrée à l'énoncé des résultats de cette thèse. Enfin, nous exposons la liste des travaux rassemblés dans cette thèse dans la Section 6, et nous terminons par plusieurs perspectives (Section 7).

1. EQUATION DE SCHRÖDINGER EN PHYSIQUE

1.1. Mécanique quantique

L'équation de Schrödinger est une équation aux dérivées partielles issue de la mécanique quantique. Le développement de ce domaine de la physique théorique fut initié à partir du début du XX^e siècle suite aux échecs de la physique classique sur plusieurs problèmes, en particulier celui du corps noir. Il fut en effet montré expérimentalement que le rayonnement d'un corps ne dépend pas de la nature de celui-ci, mais seulement de sa température, ce qui entrait en contradiction avec les considérations de la physique classique. En 1900, Max Planck retrouva « dans un acte désespéré » cette propriété en supposant que la matière ne peut échanger de l'énergie avec le rayonnement électromagnétique que par « paquets » d'énergie. Einstein reprendra cet argument 5 ans plus tard en postulant que la lumière est composée de grains élémentaires (appelés photons) pour interpréter l'effet photoélectrique. Cette dualité fut par la suite étendue à toute particule par Louis de Broglie en 1924, et confirmée par l'expérience de Davisson-Germer en 1927.

La mécanique quantique abandonne le déterminisme de la mécanique classique en faisant les postulats suivants :

— **Postulat 1** : À tout état physique d'un système, on associe un vecteur d'un espace de Hilbert $\mathcal{H}$ lié à un ensemble complet d'observables compatibles.

— **Postulat 2** : Pour un état physique caractérisé par un vecteur $|\psi\rangle \in \mathcal{H}$, la probabilité de trouver $\{a_1, ..., a_N\}$ durant la mesure de l'ensemble des grandeurs $A_1, ..., A_N$ est indépendante de l'ordre dans lequel les mesures sont faites et vaut

$$\mathbb{P}(a_1, ..., a_N) = |\langle\psi|a_1, ..., a_N\rangle|^2.$$

Après la mesure, le système est dans l'état $|a_1, ..., a_N\rangle$.

— **Postulat 3** : L'application F_{t,t_0} liée au flot (de sorte que $|\psi(t)\rangle = F_{t,t_0}|\psi(t_0)\rangle$) conserve le produit scalaire.

Ces hypothèses ont permis à Erwin Schrödinger d'obtenir en 1925 une équation générale d'évolution sur $|\psi(t)\rangle$:

$$i\hbar\frac{\mathrm{d}|\psi\rangle}{\mathrm{d}t} = \hat{H}(t)|\psi(t)\rangle,$$

où $\hat{H}$ est l'opérateur énergie. L'énergie classique pour une particule soumise à un potentiel $V(x)$ étant $E = \frac{1}{2}m|v|^2 + V(x) = \frac{|p|^2}{2m} + V(x)$ où $p = mv$ est l'impulsion de la particule, $\hat{H}$ pour une telle particule s'écrit donc sous la forme $\hat{H} = \frac{1}{2m}\hat{P}^*\,\hat{P} + V(\hat{X})$ avec $\hat{X}$ (resp. $\hat{P}$) l'opérateur position (resp. impulsion). Via des considérations sur les groupes de translation, ces derniers sont reliés par la relation de commutation canonique $[\hat{X}_j, \hat{P}_k] = i\hbar\delta_{jk}$ et ne peuvent donc être compatibles. Cependant, le théorème de Stone-von Neumann permet de conclure que, à isomorphisme près, $\mathcal{H} = L^2(\mathbb{R}^d_x, \mathbb{C})$ avec $\hat{X} = x\,\cdot$ (l'opérateur multiplication par x) et $\hat{P} = \frac{\hbar}{i}\nabla$, de sorte que $\hat{P}^*\,\hat{P} = -\hbar^2\Delta$, permettant d'obtenir l'équation de Schrödinger linéaire avec potentiel portant sur la fonction d'onde $\psi = \psi(t, x)$:

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\Delta\psi = V(x)\psi. \tag{1.1}$$

1.2. Du linéaire au non-linéaire

L'équation (1.1) est une équation caractérisant la fonction d'onde d'une seule particule soumise à un potentiel V. Pour traiter le cas de plusieurs particules, il est alors nécessaire de tensoriser les différents espaces de Hilbert précédents : la fonction d'onde du système entier $\psi = \psi(t, x_1, ..., x_N)$ (pour N particules) est alors dans l'espace $L^2((\mathbb{R}^d_x)^N, \mathbb{C})$ pour tout temps $t \in \mathbb{R}$. Dans le cadre de particules bosoniques, cette fonction d'onde est également symétrique en les variables d'espace. De plus, si le potentiel est issu d'interactions entre les particules, l'opérateur énergie s'écrit alors :

$$\hat{H}_N\psi = -\frac{1}{2m}\hbar^2 \sum_{j=1}^{N} \Delta_{x_j}\psi + \frac{1}{N} \sum_{1\leq j<k\leq N} r^{-d} V\Big(\frac{x_j - x_k}{r}\Big)\psi,$$

où $r = N^{-\beta}$ ($\beta > 0$) est une longueur typique d'interaction. Dans le cadre de la limite de champ moyen $N \to \infty$ et $r \to 0$, il est alors pratique d'étudier l'évolution de $\gamma_{t,N} \in \mathcal{L}(L^2(\mathbb{R}^{d\times N}))$ la projection orthogonale sur $\psi(t)$, ou de manière équivalente son noyau

$$\tilde{\gamma}_{t,N}(x_1, ..., x_N, y_1, ..., y_N) = \overline{\psi(t, y_1, ..., y_N)}\psi(t, x_1, ..., x_N).$$

Il est alors connu que, pour une donnée initiale du type $\psi(0, x_1, ..., x_N) = \prod_{k=1}^N \phi(x_k)$ où $\phi \in H^1(\mathbb{R}^d)$ est fixé, les distributions marginales associées $\tilde{\gamma}^{(k)}_{t,N}$ convergent (en un certain sens) vers

$$\tilde{\gamma}^{(k)}_t(x_1, ..., x_k, y_1, ..., y_k) = \prod_{j=1}^{k} \overline{\phi_t(y_j)}\phi_t(x_j),$$

où $\phi_t = \phi_t(x)$ est solution de l'équation de Schrödinger non-linéaire cubique

$$i\hbar\partial_t\phi_t + \frac{\hbar^2}{2m}\Delta\phi_t = b_0|\phi_t|^2\phi_t, \tag{1.2}$$

où $b_0 = \int_{\mathbb{R}^d} V(x)\,\mathrm{d}x$, avec donnée initiale $\phi_0 = \phi$ (voir par exemple [66] en dimension $d = 3$).

Cette équation (1.2) apparait également comme une équation d'enveloppe pour la propagation des vagues d'eau satisfaisant l'équation de Korteweg-de Vries (dit KdV) :

$$\partial_t u + \partial_x^3 u + \varepsilon u \partial_x u = 0,$$

dans le cadre où $\varepsilon \to 0$ (voir [130, 131]), ou encore pour les équations de Maxwell en électrodynamique (voir [104] par exemple). Pour une liste d'exemples plus exhaustive, on se référera à [138].

2. Equation de Schrödinger et non-linéarité de type puissance

2.1. Semi-groupe de Schrödinger, estimées de Strichartz et application au problème non-linéaire

L'apparition de ces équations de Schrödinger non-linéaires (appelées NLS)

$$i\partial_t u + \frac{1}{2}\Delta u + f(|u|^2)u = 0 \tag{2.1}$$

(pour f une fonction $\mathcal{C}^\infty([0, \infty))$ à valeurs réelles et où l'on a renormalisé les termes) dans des modèles et approximations physiques conduit donc à l'étude mathématique de celles-ci. L'étude du problème de Cauchy est naturellement la première étape. Pour cela, une première approche consiste à utiliser les propriétés du problème linéaire. Notons $S(t)u_0$ la solution de l'équation de Schrödinger linéaire

$$i\partial_t u + \frac{1}{2}\Delta u = 0, \tag{2.2}$$

de donnée initiale $u(0) = u_0$. On remarquera tout d'abord que ces deux équations, aussi bien linéaire que non-linéaire, ont plusieurs invariants classiques :

1. L'invariance par translation en temps et en espace,
2. L'invariance par rotation,
3. L'invariance galiléenne : si $u(t, x)$ est solution de (2.1) (resp. (2.2)), alors $u(t, x - vt)e^{-i\left(v\cdot x - t\frac{|v|^2}{2}\right)}$ l'est également,
4. L'invariance de jauge (multiplication par une constante de module 1).

De plus, lorsque la non-linéarité est de type puissance, (2.1) vérifie également une invariance d'échelle : si $u(t,x)$ est solution de (2.1) avec $f(x) = \pm x^{\frac{\alpha-1}{2}}$, alors $\lambda^{\frac{1}{\alpha-1}} u(\lambda t, \lambda^{\frac{1}{2}} x)$ (pour $\lambda > 0$) l'est également. En outre, $-\Delta$ étant un opérateur auto-adjoint défini positif de domaine $H^2(\mathbb{R}^d)$ dense dans $L^2(\mathbb{R}^d)$, il est connu que $S(t)$ est un semi-groupe de contraction unitaire fortement continu, noté $e^{it\frac{\Delta}{2}}$, défini sur $L^2(\mathbb{R}^d)$. Ce semi-groupe vérifie des estimations de dispersion dans L^q pour $q > 2$, à savoir :

Proposition 2.1. *Pour tout $p \in [1,2]$, il existe $C_p > 0$ tel que pour tout $u_0 \in L^p(\mathbb{R}^d)$ et tout $t \neq 0$,*

$$\left\| e^{it\Delta} u_0 \right\|_{L^{p'}} \leq \frac{C_p}{|t|^{(\frac{1}{p}-\frac{1}{p'})\frac{d}{2}}} \|u_0\|_{L^p}.$$

Cette première propriété permet d'obtenir d'autres estimations $L_t^p L_x^q$ appelées estimations de Strichartz. Pour cela, on dit que (p,q) est un couple (Strichartz) admissible dans $\mathbb{R}^d$ si $p, q \geq 2$ (possiblement infini), $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$ et $(p,q,d) \neq (2, \infty, 2)$. De plus, pour tout $p \geq 1$, on note $p' \geq 1$ vérifiant $\frac{1}{p} + \frac{1}{p'} = 1$.

Lemme 2.2 (Estimations de Strichartz, [137, 76, 160])**.** *Pour toutes paires (p,q) et $(\tilde{p}, \tilde{q})$ admissibles, il existe une constante $C > 0$ tel que pour tout $u_0 \in L^2(\mathbb{R}^d)$ et $F = F(t,x) \in L_t^{\tilde{p}'} L_x^{\tilde{q}'}$, on a les estimations suivantes :*

$$\left\| e^{it\Delta} u_0 \right\|_{L_t^p L_x^q} \leq C \|u_0\|_{L^2},$$

$$\left\| \int_{\mathbb{R}^d} e^{-is\Delta} F(s,.) \,\mathrm{d}s \right\|_{L^2} \leq C \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}},$$

$$\left\| \int_{\mathbb{R}^d} e^{i(t-s)\Delta} F(s,.) \,\mathrm{d}s \right\|_{L_t^p L_x^q} \leq C \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}}.$$

En outre, (2.1) (avec donnée initiale $u(0) = u_0$) peut être réécrit via une formule de Duhamel :

$$u(t) = e^{it\frac{\Delta}{2}} u_0 + i \int_0^t e^{i(t-s)\frac{\Delta}{2}} \Big(f(|u(s)|^2) u(s) \Big) \,\mathrm{d}s.$$

Pour simplifier, prenons une non-linéarité de type puissance en supposant que $f(x) = \pm x^{\frac{\alpha-1}{2}}$ pour un certain $\alpha \in (1, \infty)$. Les estimations de Strichartz précédentes associées à la formulation de Duhamel permettent d'obtenir une théorie de Cauchy locale, par exemple :

- pour une donnée initiale H^s, dans $\mathcal{C}_t H_x^s$ pour $s > \frac{d}{2}$ si $\frac{\alpha-1}{2}$ est de plus entier,
- pour une donnée initiale L^2, dans $\mathcal{C}_t L_x^2 \cap L_t^r L_x^{\alpha+1}$ lorsque $1 < \alpha < 1 + \frac{4}{d}$ (cas L^2 sous-critique) ou $\alpha = 1 + \frac{4}{d}$ (cas L^2 critique), où
$$r = \frac{4(\alpha+1)}{d(\alpha-1)},$$
- pour une donnée initiale H^1, dans $\mathcal{C}_t H_x^1 \cap L_t^r W_x^{1,\rho}$ lorsque $1 < \alpha < 1 + \frac{4}{d-2}$ si $d \geq 3$ ou $\alpha < \infty$ si $d \leq 2$ (cas H^1 sous-critique), avec
$$(r, \rho) = \begin{cases} \Big(\frac{4(\alpha+1)}{(d-2)(\alpha-1)}, \frac{d(\alpha+1)}{d+\alpha-1} \Big), & \text{si } d \geq 3, \\ \Big(\frac{4(\alpha+1)}{d(\alpha-1)}, \alpha+1 \Big), & \text{si } d \leq 2. \end{cases}$$

Des formulations plus précises pourront être trouvées dans [39, 41, 139] par exemple.

2.2. Masse, énergie et solutions globales

La suite logique est alors la question de l'existence de solutions globales. Ce problème est fortement lié à certaines quantités conservées par le flot de cette équation non-linéaire. En effet, dans le cadre de l'équation linéaire avec potentiel (1.1), sont conservés :

- la masse $M(u(t)) \coloneqq \int_{\mathbb{R}^d} |u(t,x)|^2 \,\mathrm{d}x$,
- le moment angulaire $\mathcal{J}(u(t)) \coloneqq \operatorname{Im} \int_{\mathbb{R}^d} \nabla u(t,x) \overline{u(t,x)} \,\mathrm{d}x$, lorsque V ne dépend pas de x,
- l'énergie $E_V(u(t)) \coloneqq \frac{\hbar^2}{2} \int_{\mathbb{R}^d} |\nabla u(t,x)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^d} V(x) |u(t,x)|^2 \,\mathrm{d}x$.

Dans le cadre non-linéaire, le flot de (2.1) conserve lui aussi la masse et le moment angulaire (dès lors que V ne dépend pas de x pour cette dernière), tandis que l'énergie est modifiée via la relation :

$$\mathcal{E}_f(u(t)) \coloneqq \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t,x)|^2 \,\mathrm{d}x - \int_{\mathbb{R}^d} F(|u(t,x)|^2) \,\mathrm{d}x, \tag{2.3}$$

où $F(a) = \int_0^a f(y)\,\mathrm{d}y$.

La théorie de Cauchy locale dans le cas L^2 sous-critique associée au fait que la masse soit conservée, autrement dit la norme $L^2(\mathbb{R}^d)$, donne alors une théorie de Cauchy globale. Ceci n'est cependant pas le cas dans le cas L^2 critique, dû au fait que le temps d'existence minimal dépend complètement de la donnée initiale, contrairement au cas sous-critique où il ne dépend que de la norme L^2 de la donnée initiale.

Similairement, l'énergie est liée à l'existence de solutions globales au niveau H^1, l'exemple le plus simple étant le cas où la fonction F est négative : la norme $\dot{H}^1$ est alors bornée grâce à l'énergie tandis que la norme L^2 reste constante via la masse. Cependant, ce n'est pas le seul cas où on peut conclure. Reprenons notre exemple d'une non-linéarité de type puissance. Si $f(x) = -x^{\frac{\alpha-1}{2}}$, on a donc une solution globale quelle que soit la donnée initiale. A l'inverse, si $f(x) = x^{\frac{\alpha-1}{2}}$, il est alors nécessaire d'utiliser l'inégalité de Gagliardo-Nirenberg pour obtenir (sous certaines conditions) une borne de l'énergie potentielle $(-\int_{\mathbb{R}^d} F(|u(t,x)|^2)\,\mathrm{d}x)$ en fonction de la norme H^1 :

Lemme 2.3 (Inégalité de Gagliardo-Nirenberg)**.** *Pour tout $2 \leq p < 2^*$ où $2^* = \frac{2d}{d-2}$ si $d \geq 3$ et $2^* = \infty$ sinon, il existe une constante $C_{\mathrm{GN},p} > 0$ tel que pour tout $u \in H^1(\mathbb{R}^d)$,*

$$\|u\|_{L^p} \leq C_{\mathrm{GN},p} \|\nabla u\|_{L^2}^{\theta} \|u\|_{L^2}^{1-\theta},$$

où $0 \leq \theta \leq 1$ est tel que $\frac{1}{p} = (\frac{1}{2} - \frac{1}{d})\theta + \frac{1-\theta}{2}$.

Cette inégalité permet d'obtenir une solution globale H^1 lorsque $f(x) = x^{\frac{\alpha-1}{2}}$ dans les cas suivants :

— $\alpha < 1 + \frac{4}{d}$,

— $\alpha = 1 + \frac{4}{d}$ et $\|u_0\|_{L^2}^{\frac{4}{d}} < \frac{\alpha+1}{2C_{\mathrm{GN},\alpha+1}}$,

— $1 + \frac{4}{d} < \alpha < 1 + \frac{4}{d-2}$ et $\|u_0\|_{H^1}$ est assez petit.

Ces cas sont optimaux, dans le sens où, par exemple, il existe une solution de donnée initiale $\|u_0\|_{L^2}^{\frac{4}{d}} = \frac{\alpha+1}{2C_{\mathrm{GN},\alpha+1}}$ qui explose en temps fini dans le cas $\alpha = 1 + \frac{4}{d}$. De nombreux autres résultats de ce type peuvent être trouvés dans [150].

2.3. Comportement en temps long

Nous nous plaçons dorénavant dans un cadre où nous avons une solution globale en temps. Immédiatement vient alors la question de l'asymptotique en temps long : la solution tend-elle vers une fonction limite lorsque $t \to +\infty$? Si oui, à quelle vitesse ? La réponse à cette double question dépend fortement de la non-linéarité, et plus particulièrement de ses variations près de 0. De façon heuristique, la solution va minimiser l'énergie potentielle au cours du temps. Ainsi, si 0 est un maximum de F, il est logique de penser que la solution devrait tendre vers 0, tandis que celle-ci devrait au contraire s'accroître (au moins localement) s'il s'agit d'un minimum (local) de F. Dans le cas d'une non-linéarité de type puissance comme précédemment, le type de réponse dépend donc de son signe. Cependant, il ne faut pas oublier que la norme L^2 est conservée pour ces solutions, la minimisation de l'énergie potentielle se fait donc à norme L^2 constante. En particulier, si la solution tend vers 0 en un certain sens, cela ne peut être en norme L^2.

Dispersion et scattering. Dans le cadre linéaire, les solutions de (2.2) vérifient la Proposition 2.1. En particulier, lorsque la donnée initiale est intégrable sur $\mathbb{R}^d$, $e^{it\Delta}u_0$ devient immédiatement L^∞ et décroit comme $|t|^{-\frac{d}{2}}$. Un comportement similaire est également attendu dans le cadre non-linéaire lorsque la non-linéarité ne s'oppose pas à cette dispersion. Dans notre exemple d'une non-linéarité de type puissance, il s'agit alors du cas où $f(x) = -x^{\frac{\alpha-1}{2}}$: on parle alors d'un régime défocalisant.

Lorsque la non-linéarité est assez "plate" en 0 (c'est-à-dire α assez grand), et puisque la solution tend heuristiquement vers 0, la non-linéarité peut alors être considérée comme "négligeable" en temps grand et on s'attend à retrouver la même vitesse de dispersion (en $|t|^{\frac{d}{2}}$) que dans le cas linéaire. Il est cependant nécessaire d'avoir α assez grand pour que la non-linéarité soit réellement néligeable. En effet, une autre façon de voir est d'injecter cette vitesse de dispersion dans le "potentiel non-linéaire" $f(|u|^2)$, c'est-à-dire de considérer $f(t^{-d})$: si cette fonction est intégrable pour $t \to \infty$, ce qui correspond à $\alpha > 1 + \frac{2}{d}$ lorsque f est de type puissance, on peut espérer que le terme non-linéaire soit effectivement assez négligeable en temps grand pour que la partie linéaire dirige asymptotiquement le flot de l'équation non-linéaire.

Ces considérations sont plus rigoureusement formulés dans le cadre de la théorie du *scattering*. On dit que le comportement de la solution de (2.1) est asymptotiquement linéaire si il existe une fonction $\psi_+ \in \Sigma$ (pour un certain espace fonctionnel Σ) telle que

$$\left\| e^{-i\frac{t}{2}\Delta}\psi(t) - \psi_+ \right\|_{\Sigma} \underset{t\to\infty}{\longrightarrow} 0.$$

Cette théorie a fait l'objet de nombreuses études. Il est par exemple connu que, pour une non-linéarité de type puissance, cette conjecture est vraie lorsque $\alpha \geq \alpha_0(d)$ pour $\Sigma = H^1(\mathbb{R}^d) \cap \mathcal{F}(H^1(\mathbb{R}^d))$, où $\alpha_0(d)$ est l'exposant de Strauss et vérifie $1 + \frac{2}{d} < \alpha_0(d) \leq 1 + \frac{4}{d}$ (voir par exemple [39]). Une convergence plus faible, dans $L^2(\mathbb{R}^d)$, est également prouvé pour $\alpha > 1 + \frac{2}{d}$

dans [144] avec $\psi_+ \in L^2(\mathbb{R}^d)$. Cependant, pour $1 < \alpha \leq 1 + \frac{2}{d}$, des effets à longue portée sont observés et les dynamiques linéaires et non-linéaires ne peuvent plus être comparées de la même manière (voir [94] et ses références). Il a cependant été récemment montré que ces effets à longue portée n'induisent qu'une modification de la phase à l'ordre dominant, sans changer le caractère dispersif ni la vitesse de dispersion ([29]).

Soliton, multi-solitons et Resolution Conjecture. Dans le cas contraire d'une non-linéarité de la forme $f(x) = x^{\frac{\alpha-1}{2}}$, l'énergie potentielle sera minimisée lorsque u prend moralement de grandes valeurs, la solution aurait donc heuristiquement tendance à croître (en valeur absolue), tout du moins localement puisque restant de norme L^2 constante : il s'agit alors du régime focalisant. Cependant, cette croissance est également compensée par la borne de la norme $\dot{H}^1$ via l'énergie lorsque α n'est pas trop grand, de la même manière que pour obtenir des solutions globales. De fait, des solutions stationnaires non triviales pour de telles équations, de la forme

$$u(t,x) = e^{i\omega t}\psi_\omega(x), \tag{2.4}$$

existent si et seulement si $\omega > 0$ et $r_0 = \inf\{r > 0 |\, F(r) = \omega r\}$ existe et satisfait $f(r_0) > \omega$ ([19, 17]). $\psi_\omega \in H^1$ est alors une fonction réelle solution de

$$\frac{1}{2}\Delta\psi_\omega + f(|\psi_\omega|^2)\psi_\omega = \omega\, \psi_\omega.$$

Pour cette équation, sous la même hypothèse, il existe une unique solution radiale réelle positive, notée Q_ω. Cette fonction est l'unique minimiseur (aux invariances 1 et 4 près) de l'énergie $\mathcal{E}_f(\psi)$ (2.3) sous la condition $\|\psi\|_{L^2} = \|Q_\omega\|_{L^2}$. Ces solutions particulières $e^{i\omega t}Q_\omega(x)$ de (2.1), appelées ondes solitaires ou solitons, sont fondamentales dans la dynamique de l'équation. En effet, sous certaines hypothèses, elles sont connues comme étant stables, dans le sens de la stabilité orbitale :

Définition 2.4. *On dit qu'une onde solitaire* (2.4) *est (orbitalement)* H^1*-stable si pour tout* $\varepsilon > 0$*, il existe* $\delta > 0$ *tel que pour tout* $u_0 \in H^1(\mathbb{R}^d)$ *vérifiant*

$$\inf_{x_0,v_0,\gamma_0}\left\|u_0(x) - \psi_\omega(x-x_0)e^{i\left(v_0\cdot x+\gamma_0\right)}\right\|_{H^1} < \delta,$$

la solution $u(t)$ *de* (2.1) *avec donnée initiale* $u(0) = u_0$ *satisfait*

$$\sup_{t\geq 0}\inf_{x_1,v,\gamma}\left\|u(t,x) - \psi_\omega(x-x_1)e^{i\left(v\cdot x+\gamma\right)}\right\|_{H^1} < \varepsilon.$$

En 1982, Cazenave et Lions [42] ont prouvé la stabilité orbitale pour Q_ω sous hypothèse de compacité sur les suites minimisantes pour l'énergie en utilisant les méthodes de concentration compacité développées par Lions [109]. En 1986, par une méthode basée sur un développement des quantités conservées autour des ondes solitaires, Weinstein [152] a également montré cette stabilité pour Q_ω sous des hypothèses sur le spectre de l'opérateur linéarisé autour de Q_ω et de non-dégénérescence :

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\|Q_\omega\|^2_{L^2(\mathbb{R}^d)}|_{\omega=\omega_0} > 0,$$

montrant également que ces hypothèses sont vérifiées dans le cas d'une puissance sous-critique pour $d = 1$ ou 3 (ou sous d'autres conditions moins restrictives). Inversement, si

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\|Q_\omega\|^2_{L^2(\mathbb{R}^d)}|_{\omega=\omega_0} < 0,$$

Q_ω est instable dans H^1 ([83]).

Une autre question importante concerne la compréhension des interactions entre ces solitons, afin de mieux comprendre la dynamique et les comportements induits par l'équation. Cette question se pose d'ailleurs plus généralement dans le contexte d'équations dispersives non-linéaires. En particulier, il a été observé et prouvé que, pour une large classe de données initiales, les solutions de l'équation de Korteweg-de Vries (KdV) sont globales et se décomposent en temps long comme une somme finie de solitons plus un terme dispersif ([64, 132]). Ce comportement, qui semble être générique dans le cadre d'équations dispersives non-linéaires, amène à une conjecture appelée *Conjecture de Résolution (en Soliton)* (*(Soliton) Resolution Conjecture* en anglais) qui, formulée vaguement, établit que toute solution globale d'une équation dispersive non-linéaire (focalisante) se décompose en temps grand comme une combinaison de structures non dispersives (dont font partie les sommes de solitons) et d'un terme radiatif.

L'un des sujets usuels associés à cette question est l'existence de multi-solitons, solutions de (2.1) convergeant (en un certain sens) vers une somme de solitons quand $t \to \infty$, et leur stabilité (dans le sens de stabilité asymptotique quand $t \to \infty$). Concernant NLS, la première méthode utilisée pour construire des multi-solitons fut la transformée de scattering inverse ([161]). Cette méthode permet également de prouver la Conjecture de Résolution, mais est restreinte aux équations complètement intégrables, dont font partie KdV et NLS cubique en dimension 1. En particulier, la preuve de cette conjecture dans le cas KdV cité plus haut utilise fortement le caractère complètement intégrable de cette équation. Si certaines équations non-intégrables comme l'équation des ondes en régime critique ([63]) ou wave maps ([52]) ont été traitées, cette conjecture reste un problème ouvert pour la plupart d'entre elles.

Une autre méthode pour construire des multi-solitons pour des équations non-intégrables a été introduite en 1990 par Merle [119] pour NLS L^2 critique. Cette méthode utilise des techniques d'énergie, dans le sens où la variation seconde de l'énergie est utilisée comme une fonctionnelle de Lyapunov pour contrôler la différence d'une solution avec la somme de solitons considérée. Cette méthode fut étendue par la suite à l'équation de Korteweg-de Vries généralisée (gKDV) ([115]) et à NLS L^2 sous-critique ([116]), mais fut également modifiée pour être appliquée au cas de L^2 sur-critique ([55]), ou encore à des profils composés d'états excités ([53]). Citons également (de manière non-exhaustive) quelques travaux sur l'équation de Klein-Gordon non-linéaire ([56, 54, 16]).

Quant à la stabilité de ces solutions, il est naturel d'attendre qu'un multi-soliton soit asymptotiquement stable dans la situation où les interactions sont locales et où les solitons sont stables et exponentiellement décroissants. Pour quelques équations comme gKdV ou NLS, cette stabilité est montrée sous l'hypothèse d'une non-linéarité assez plate en 0, et d'une vitesse relative entre les différents solitons assez grande pour NLS (voir par exemple [117, 118] et leurs références). Inversement, dès que l'un des solitons qui composent la somme de solitons est instable, le multi-soliton construit devrait *a priori* être instable, et de nombreux résultats, complets ou partiels, abondent dans ce sens ([53, 43, 82, 125, 81, 100, 123, 124]). Notons également qu'une description plus précise de cette stabilité existe pour KdV, avec également des solutions explicites particulières appelées N-solitons ([122, 117]).

2.4. Analyse semi-classique

En revenant à la dérivation physique de NLS (1.2), les constantes physiques impliquées dans l'équation sont très petites comparées aux ordres de grandeur de la physique classique : la constante de Planck réduite vaut environ $\hbar \sim 1,05.10^{-34} J.s$. Ainsi, on introduit mathématiquement une constante semi-classique dans NLS :

$$i\varepsilon\partial_t u_\varepsilon + \frac{\varepsilon^2}{2}\Delta u_\varepsilon + f(|u_\varepsilon|^2)u_\varepsilon = 0, \tag{2.5}$$

et il devient intéressant de se poser la question du comportement lorsque ε tend vers 0. Considérer directement la limite de u_ε (en un certain sens) serait *a priori* trop naïf : il est physiquement attendu que des oscillations de l'ordre de $\hbar$ pour (1.2), c'est-à-dire de l'ordre de ε pour (2.5), apparaissent instantanément. Les outils et méthodes développés prennent cependant en compte ces fortes oscillations dans l'asymptotique $\varepsilon \to 0$. Nous présenterons ici deux méthodes : la transformée de Wigner et l'analyse BKW.

Transformée et mesure de Wigner. La transformée de Wigner fut introduite par Wigner en 1932 ([154]). Elle est définie pour n'importe quelle suite de fonctions $f_\varepsilon = f_\varepsilon(x) \in L^2(\mathbb{R}^d)$ (pour $\varepsilon > 0$) par

$$W_\varepsilon(x,\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\xi\cdot z} f_\varepsilon\left(x + \frac{\varepsilon z}{2}\right)\overline{f_\varepsilon\left(x - \frac{\varepsilon z}{2}\right)}\,\mathrm{d}z, \qquad (x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d.$$

W_ε est alors une fonction définie sur l'espace des phases à valeurs réelles, mais qui peut être non-intégrable ou négative. Une légère modification de cette fonction, appelée la transformée de Husimi, définie par

$$W_\varepsilon^H = W_\varepsilon * G_\varepsilon,$$

où

$$\gamma_\varepsilon(x) = \frac{1}{(\pi\varepsilon)^{\frac{d}{2}}}\exp\left(-\frac{|x|^2}{\varepsilon}\right), \qquad G_\varepsilon(x,\xi) = \gamma_\varepsilon(x)\,\gamma_\varepsilon(\xi), \qquad \text{pour } x,\xi\in\mathbb{R}^d,$$

contourne ces problèmes : la transformée de Husimi est positive et intégrable ([110]). En supposant f_ε uniformément borné dans $L^2(\mathbb{R}^d)$, il est connu qu'une sous-suite de W_ε et W_ε^H converge dans le dual de

$$\mathcal{A} := \left\{\phi\in\mathcal{C}_0(\mathbb{R}^d_x\times\mathbb{R}^d_\xi), (\mathcal{F}_\xi\phi)(x,z)\in L^1(\mathbb{R}^d_z,\mathcal{C}_0(\mathbb{R}^d_x))\right\},$$

avec la topologie faible-$*$, vers une mesure W positive, appelée mesure de Wigner ([110, 88]). Les suites de fonctions $(f_\varepsilon)_{\varepsilon>0}$ à considérer pour obtenir une mesure de Wigner non triviale doivent par ailleurs être ε-oscillantes. Cette considération est bien adaptée dans le cas où f_ε est remplacée par $u_\varepsilon(t)$ solution de l'équation de Schrödinger linéaire avec potentiel $i\varepsilon\,\partial_t u_\varepsilon + \frac{\varepsilon^2}{2}\Delta u_\varepsilon = V u_\varepsilon$ avec donnée initiale adaptée (typiquement du type BKW $u_{0,\varepsilon} = \rho(x)e^{i\frac{\phi_\varepsilon}{\varepsilon}}$ avec $(1+|x|)\rho\in L^2(\mathbb{R}^d)$, ϕ à valeurs réelles et $\nabla\rho$ et $\rho\nabla\phi$ dans L^2) : si le potentiel V est assez lisse, la mesure de Wigner qui en résulte (qui dépend donc également du temps) satisfait l'équation cinétique (ou équation de Vlasov)

$$\partial_t W + \xi\cdot\nabla_x W + \nabla_x V\cdot\nabla_\xi W = 0.$$

Cette équation cinétique est fondamentale en théorie des gaz au niveau mésoscopique : la transformée de Wigner fait donc bien le lien entre physique quantique et classique dans ce cadre. Ces considérations peuvent également s'étendre au cas non-linéaire, on citera par exemple le cas où le potentiel s'écrit comme une convolution $V = V_0 * \rho$ sous réserve d'hypothèses sur V_0 ([110]). Cet exemple n'est pas le seul, et cet outil a été intensivement étudié (voir par exemple [110, 10, 88, 87]), même s'il a également été observé des limitations à cette approche ([32] pour une liste d'exemples).

Analyse BKW. Une autre approche consiste à trouver une asymptotique complète à la fonction d'onde. Pour cela, l'asymptotique à chercher doit forcément prendre en compte les fortes oscillations à la limite $\varepsilon \to 0$. Plus précisément, le but est souvent de chercher des solutions à l'équation (2.5) sous la forme

$$u(t,x) = \sqrt{\rho^\varepsilon(t,x)}e^{i\frac{\phi^\varepsilon(t,x)}{\varepsilon}}, \tag{2.6}$$

où ρ^ε et ϕ^ε convergent (en un certain sens) à la limite $\varepsilon \to 0$. On remarque que ces fonctions d'onde ont bien des oscillations en ε^{-1}. Ces fonctions ρ^ε et ϕ^ε peuvent être considérées comme des fonctions réelles, auquel cas celles-ci doivent vérifier le système

$$\begin{cases} \partial_t\phi^\varepsilon + \dfrac{1}{2}\nabla\phi^\varepsilon\cdot\nabla\phi^\varepsilon + f(\rho_\varepsilon) = \dfrac{\varepsilon^2}{2}\dfrac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}, \\ \partial_t\rho^\varepsilon + \nabla\phi^\varepsilon\cdot\nabla\rho^\varepsilon + \rho^\varepsilon\Delta\phi^\varepsilon = 0. \end{cases}$$

En considérant $v^\varepsilon = \nabla\phi^\varepsilon$, on obtient un système appelé *forme hydrodynamique* de NLS :

$$\begin{cases} \partial_t v^\varepsilon + (v^\varepsilon\cdot\nabla)v^\varepsilon + \nabla f(\rho_\varepsilon) = \dfrac{\varepsilon^2}{2}\nabla\left(\dfrac{\Delta\sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}}\right), \\ \partial_t\rho^\varepsilon + \operatorname{div}(\rho^\varepsilon v^\varepsilon) = 0. \end{cases}$$

Formellement, le cas $\varepsilon = 0$ correspond à une équation d'Euler compressible. Dans le cas $\varepsilon > 0$, le terme source additionnel est appelé *pression quantique*, et correspond en mécanique des fluides à un terme de Korteweg modélisant les effets de capillarité. Plusieurs outils ont été développés dans ce sens, parmi lesquels la transformée de Madelung ([111, 103]), permettant une étude des propriétés qualitatives des solutions de NLS avec des conditions de bord non nulles à l'infini, de type *Gross-Pitaevskii*, lorsque la solution ne s'annule pas "trop souvent".

Cependant, cette pression quantique est problématique dans le cadre de données L^2 par exemple, de par la division par $\sqrt{\rho^\varepsilon}$. Pour éviter ce problème, Grenier ([79]) a remplacé $\sqrt{\rho^\varepsilon}$ par une fonction a^ε pouvant être complexe, laissant un degré de liberté fixé par le fait que le couple $(a^\varepsilon, \phi^\varepsilon)$ doit alors vérifier

$$\begin{cases} \partial_t\phi^\varepsilon + \dfrac{1}{2}\nabla\phi^\varepsilon\cdot\nabla\phi^\varepsilon + f(|a^\varepsilon|^2) = 0, \\ \partial_t a^\varepsilon + \nabla\phi^\varepsilon\cdot\nabla a^\varepsilon + \dfrac{1}{2}a^\varepsilon\Delta\phi^\varepsilon = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon. \end{cases} \tag{2.7}$$

Lorsque ce système d'équations est satisfait, $u^\varepsilon = a^\varepsilon e^{i\frac{\phi^\varepsilon}{\varepsilon}}$ est alors bien solution de (2.5). Ce choix est alors beaucoup plus robuste que la transformée de Madelung dans la limite semi-classique ([31, 3]). De plus, de la même manière que précédemment, en considérant $v^\varepsilon = \nabla\phi^\varepsilon$, on obtient :

$$\begin{cases} \partial_t v^\varepsilon + (v^\varepsilon\cdot\nabla)v^\varepsilon + \nabla\Big(f(|a^\varepsilon|^2)\Big) = 0, \\ \partial_t a^\varepsilon + v^\varepsilon\cdot\nabla a^\varepsilon + \dfrac{1}{2}a^\varepsilon\operatorname{div} v^\varepsilon = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon. \end{cases}$$

La limite formelle de ce système lorsque $\varepsilon \to 0$ est la version symétrisée du système d'Euler compressible ([44, 112]), avec $a^0 = \sqrt{\psi^0}$ dès lors que a^0 est réelle (ce qui est équivalent à dire que la donnée initiale de a^0 est réelle).

Le système d'Euler compressible a par ailleurs été longuement étudié. En particulier, il est attendu que la solution à cette équation ne reste pas régulière pour tout temps même lorsque la donnée initiale l'est, et même explose en temps fini ([134, 6]). Par exemple, pour des données initiales à support compact, il a été montré une formation de singularités en temps fini ([112], voir aussi [155]).

On notera que, à partir de la connaissance de v^ε, il est alors simple de remonter à ϕ^ε en utilisant la première équation du système (2.7). En effet, on a alors explicitement, pour une donnée initiale $\phi^\varepsilon(0) = \phi^\varepsilon_{\text{in}}$,

$$\phi^\varepsilon(t,x) = \phi^\varepsilon_{\text{in}}(x) - \int_0^t \Big(\frac{1}{2}|v^\varepsilon(\tau,x)|^2 + f(|a^\varepsilon(\tau,x)|^2)\Big)\,\mathrm{d}\tau.$$

Comparaison entre ces deux méthodes. Comme expliqué précédemment, les fonctions à considérer pour la transformée de Wigner sont uniformément bornées dans L^2 et ε-oscillantes ([88]). La forme (2.6) utilisée pour l'analyse BKW en est un cas particulier. Par ailleurs, il est connu que les équations d'Euler voient généralement apparaître des caustiques en temps fini. Ces caustiques posent alors problème dans l'analyse BKW, qui ne peut donc se faire que localement en temps. Ce n'est pas le cas de l'approche de Wigner (au moins dans le cadre linéaire), qui passe à travers ces caustiques ([136]).

La transformée de Wigner peut être donc considérée comme plus générale que l'analyse BKW. Pour autant, cette dernière a de nombreux avantages. En effet, la transformée de Wigner marche certes assez bien dans le cadre linéaire, mais c'est moins le cas en non-linéaire ([32]), tandis que l'analyse BKW, bien que locale en temps, reste consistante même dans le cadre non-linéaire. Par ailleurs, l'analyse BKW donne des informations plus précises, non seulement de par la forme de la fonction d'onde prescrite, mais également par le fait qu'elle permet de donner une asymptotique complète à cette fonction d'onde ([27, 34, 2, 3]).

Pour une comparaison plus complète de ces deux méthodes, on peut se référer par exemple à [136].

3. Equation de Schrödinger logarithmique

3.1. Définition et dérivation

L'équation de Schrödinger logarithmique est une équation de type NLS (2.1) où f est un multiple de la fonction logarithme :

$$i\,\partial_t u + \frac{1}{2}\Delta u + \lambda u \ln |u|^2 = 0, \tag{3.1}$$

où $\lambda \in \mathbb{R} \setminus \{0\}$. Cette équation, que l'on notera logNLS, fut introduite par Białynicki-Birula et Mycielski [23] en 1976. Pour cela, ils se sont intéressés aux théories pour lesquelles un système composé de deux sous-systèmes indépendants vérifiant une équation de Schrödinger non-linéaire (2.1) (avec des non-linéarités possiblement différentes, disons f_1 et f_2) vérifiait lui-même une équation de type (2.1) (avec une non-linéarité f_3), une propriété appelée *séparabilité des systèmes indépendants*. En utilisant le fait que la fonction d'onde d'un tel système est le produit tensoriel des fonctions d'ondes des deux sous-systèmes, cette considération revient à trouver des fonctions f_1, f_2 et f_3 (réelles) vérifiant

$$f_1(x) + f_2(y) = f_3(xy), \qquad \forall x, y > 0.$$

Il se trouve que les seules fonctions vérifiant cette équation fonctionnelle sont du type

$$f_i(x) = \lambda \ln x + c_i, \qquad i \in \{1, ..., 3\},$$

où λ, c_1 et c_2 sont des constantes réelles arbitraires et $c_3 = c_1 + c_2$. Depuis lors, cette équation se retrouve dans de nombreuses théories physiques, allant de la mécanique des ondes à l'optique ([25, 95, 98, 102, 59, 105]).

Il est à noter que, si logNLS a formellement les mêmes invariants usuels que NLS (à savoir invariance par translation en espace et temps, invariance galiléenne et multiplication par une constante de module 1), une autre invariance d'échelle est valable pour cette équation : si $u = u(t,x)$ vérifie logNLS, alors pour tout $\kappa > 0$, $u_\kappa(t,x) = \kappa u(t,x)\, e^{2it\lambda \ln \kappa}$ vérifie également logNLS. Cette invariance est extrêmement inhabituelle dans un cadre non-linéaire, puisqu'elle implique en particulier que la taille des données n'intervient pas dans la dynamique de l'équation, contrairement aux non-linéarités vues précédemment (du type puissance par exemple). Une autre implication concerne la régularité du flot $u_0 \mapsto u(t)$ pour tout $t > 0$. En effet, en dérivant l'expression précédente en $\kappa > 0$, on obtient :

$$\frac{\mathrm{d}}{\mathrm{d}\kappa} u_\kappa = (1 + 2it\lambda) u(t,x) e^{2it\lambda \ln \kappa},$$

qui n'a aucune limite (forte ou faible) lorsque $\kappa \to 0$. Dès lors, le flot ne peut être $\mathcal{C}^1$ en 0.

La non-linéarité logarithmique se retrouve par ailleurs dans de nombreuses autres équations ayant une importance physique, que cela soit en optique quantique avec les équations de Korteweg-de Vries logarithmique ou de Kadomtsev-Petviashvili logarithmique ([99, 148, 149]), en théorie quantique des champs et en cosmologie avec l'équation de Klein-Gordon logarithmique ([15, 78, 129]), ou en sciences des matériaux avec l'équation de Cahn-Hilliard avec potentiels logarithmiques ([48, 65, 75]).

3.2. Résultats actuels

Du point de vue mathématique, la non-linéarité logarithmique ne permet pas d'appliquer les résultats généraux de la section 2. En effet, ces résultats ne peuvent s'appliquer que lorsque f est suffisamment régulière (typiquement localement lipschitzienne ou $\mathcal{C}^1$) sur $[0, +\infty[$. Or, le logarithme n'est pas défini en 0, ni lipschitzien ou même borné autour de 0. De plus, la fonction logarithme n'a pas de signe, ce qui fait qu'on ne peut *a priori* pas prédire le comportement de la solution à la seule lecture de l'énergie (conservée formellement par le flot) :

$$\mathcal{E}(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 - \lambda \int_{\mathbb{R}^d} |v(x)|^2 (\ln |v(x)|^2 - 1)\,\mathrm{d}x. \tag{3.2}$$

Les premières études mathématiques pour cette équation sont l'œuvre de T. Cazenave et A. Haraux ([40, 38]) au début des années 1980. Ce n'est que récemment que de nouveaux résultats ont vu le jour, comme par exemple [58, 33, 8, 36]. Dans cette section, nous présentons les différents résultats connus sur logNLS.

On notera cependant qu'il existe une littérature assez vaste sur la version stationnaire

$$-\Delta u + \lambda u \ln |u|^2 = \omega u,$$

où ω est une constante réelle, qui est une équation elliptique. On citera par exemple [58, 147] ainsi que les références de ces deux articles.

Problème de Cauchy. Le problème de Cauchy a tout d'abord été étudié dans [40] (voir aussi [39]) pour le cas $\lambda > 0$. Pour cela, les auteurs utilisent une méthode de monotonie et le fait que l'énergie (3.2), définie dans l'espace d'énergie

$$W(\mathbb{R}^d) := \left\{ v \in H^1(\mathbb{R}^d), |v|^2 \ln |v|^2 \in L^1(\mathbb{R}^d) \right\},$$

est bien adaptée à ce cas : la partie problématique de l'énergie potentielle $-\lambda \int_{|v|\leq 1} |v(x)|^2 \ln |v(x)|^2 \,\mathrm{d}x$ est comptée positivement dans l'énergie, tandis que l'autre partie (comptée négativement) peut être facilement estimée grâce à la norme H^1, ce qui permet d'obtenir une borne uniforme de la solution dans $W(\mathbb{R}^d)$. Les auteurs utilisent en outre une identité spécifique à la non-linéarité logarithmique, à savoir que pour tout $z, z' \in \mathbb{C}$,

$$\left| \mathrm{Im} \left[(z \log|z| - z' \log|z'|) \overline{z - z'} \right] \right| \leq |z - z'|^2. \tag{3.3}$$

Ils obtiennent grâce à cela une théorie de Cauchy globale dans l'espace d'énergie $W(\mathbb{R}^d)$.

Le cas $\lambda < 0$ a ensuite été traité par R. Carles et I. Gallagher [33]. Pour combler le manque d'estimation de la partie problématique de l'énergie potentielle, comptée négativement dans ce cas, les auteurs utilisent une propriété subsidiaire de NLS en général, à savoir la propagation des moments d'ordre plus petit que 1 pour des solutions H^1 : si la solution est bornée dans H^1 sur un intervalle $[0, T]$, alors la quantité

$$I_\alpha(t) := \int_{\mathbb{R}^d} |x|^{2\alpha} |u(t,x)|^2 \,\mathrm{d}x$$

l'est également sur $[0, T]$ dès lors que $I_\alpha(0)$ est finie. Cette quantité permet d'estimer la partie problématique de l'énergie potentielle pour logNLS, tandis que la borne $L^\infty_{\mathrm{loc}}([0,\infty), H^1(\mathbb{R}^d))$ est obtenue via une estimation grossière de $\mathrm{Im}(\partial_t \nabla u \cdot \overline{\nabla u})$. Ceci leur permet d'obtenir une théorie de Cauchy globale dans $H^1(\mathbb{R}^d) \cap \mathcal{F}(H^\alpha(\mathbb{R}^d))$ pour n'importe quel $\alpha \in]0,1]$ ([33, Theorem 1.5.]).

Par ailleurs, que $\lambda > 0$ ou $\lambda < 0$, la propriété (3.3) permet d'avoir également une estimation explicite sur la continuité du flot dans L^2 :

Lemme 3.1 ([40, Lemme 2.2.1])**.** *Pour toutes solutions u et v de* (3.1) *dans* $\mathcal{C}^0(\mathbb{R}, L^2(\mathbb{R}^d))$ *et tout $s, t \in \mathbb{R}$,*

$$\|u(s) - v(s)\|_{L^2} \leq e^{2|\lambda||t-s|} \|u(t) - v(t)\|_{L^2}.$$

Données gaussiennes. Il avait déjà été noté dès 1976 ([23]) que les données initiales gaussiennes sont des données particulières qui permettent d'obtenir des solutions explicites. En effet, pour une fonction gaussienne, la non-linéarité devient un potentiel quadratique centré comme la dite gaussienne, et il est connu que les potentiels quadratiques dans NLS conservent le caractère gaussien d'une solution ([97, 96, 89, 90]). Dès lors, il est naturel de chercher des solutions gaussiennes et d'étudier leur comportement, dans l'optique de mieux comprendre en première intuition le régime de l'équation (focalisant ou défocalisant), selon le signe de λ possiblement. On trouve alors que toute solution de donnée initiale gaussienne s'écrit (à translation en espace près) sous la forme

$$u(t,x) := b(t) \exp\left[\frac{d}{2} - x^\top A(t) x\right],$$

où A est une matrice complexe symetrique ($A^\intercal = A$) dont la partie réelle est symétrique définie positive et satisfait l'équation matricielle

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -iA^2 + 2i\lambda \,\mathrm{Re}\, A, \tag{3.4}$$

tandis que b est donnée explicitement par la connaissance de A via la relation $b(t) = b_0 \left(\det \mathrm{Re}\, A(t)\right)^{\frac{1}{4}} e^{i\phi(t)}$, où b_0 est une constante arbitraire et ϕ est explicitement donnée (voir le système d'équations (6.14), (6.15) et (6.16) de [23]). En notant $A_r(t)$ et $A_i(t)$ les parties réelles et imaginaires de $A(t)$, la conservation de l'énergie donne alors la conservation de la quantité

$$\mathrm{Tr}\Big(A_r(t) + (A_i(t)^2 A_r(t)^{-1} - \lambda \ln A_r(t)\Big).$$

Dans le cas de la dimension $d = 1$, ces relations peuvent être simplifiées : $A(t)$ est alors un nombre complexe pouvant être écrit sous la forme $A(t) = \frac{1}{2r(t)^2} - i\frac{\dot{r}(t)}{2r(t)}$, où r est une fonction réelle strictement positive satisfaisant alors l'EDO

$$\ddot{r} = \frac{1}{r^3} - \frac{2\lambda}{r}.$$

L'étude du comportement de r (c'est-à-dire de la solution de l'EDO précédente) permet de donner une intuition du régime de logNLS (3.1) et donc de mieux comprendre le comportement de la solution dans le cadre général. On notera que ces solutions en dimension 1 peuvent être tensorisées pour donner des solutions en dimension d quelconque, ce qui implique que cette étude permet de caractériser le régime en n'importe quelle dimension, celui-ci ne dépendant *a priori* donc pas de la dimension mais seulement de λ.

Cas défocalisant et dispersion. Lorsque $\lambda < 0$, dans le cas gaussien avec $d = 1$, la fonction r a été étudiée par R. Carles et I. Gallagher [33] : ils ont alors montré que la solution admet comme asymptotique pour $t \to \infty$

$$r(t) \sim 2t\sqrt{\lambda \ln t},$$

tout comme la solution de $\ddot{\tau} = -\frac{2\lambda}{\tau}$ avec $\tau(0) = 1$ et $\dot{\tau}(0) = 0$. La gaussienne s'étale donc au cours du temps, ce qui est un comportement caractéristique d'un régime dispersif. Dans la continuité, les auteurs dans le même article montrent une "dynamique universelle" pour (3.1) dès lors que la donnée initiale est dans $H^1 \cap \mathcal{F}(H^1)$: en effectuant un rescaling de la solution $u(t,x)$ par

$$u(t,x) = \frac{1}{\tau(t)^{\frac{d}{2}}} \frac{\|u_{\text{in}}\|_{L^2}}{\|\gamma\|_{L^2}} v\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot{\tau}(t)}{\tau(t)}\frac{|x|^2}{2}}, \tag{3.5}$$

où $\gamma(y) = e^{-\frac{|y|^2}{2}}$, la fonction v est alors telle que la quantité

$$\tilde{E}(t,v) = \frac{1}{\tau(t)^2}\|\nabla_y v(t)\|^2_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d} \left(|y|^2 + \left|\ln|v(t,y)|^2\right|\right) |v(t,y)|^2 \, \mathrm{d}y$$

est bornée uniformément en temps. De plus, non seulement $|v(t,.)|^2 \underset{t\to\infty}{\rightharpoonup} \gamma^2$ faiblement dans $L^1(\mathbb{R}^d)$, mais le moment de v tend également vers le moment de γ :

$$\int_{\mathbb{R}^d} |y|^2 |v(t,y)|^2 \, \mathrm{d}y \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} |y|^2 \gamma^2(y) \, \mathrm{d}y.$$

Ce résultat permet de mettre en évidence de nombreuses propriétés exceptionnelles caractérisant la dynamique de (3.1) dans ce cas. Tout d'abord, on remarque que la dispersion est inhabituelle : d'habitude en $t^{-\frac{d}{2}}$ pour l'équation de Schrödinger linéaire sans potentiel ou pour NLS avec non-linéarité de type puissance, elle est ici en $\left(t\sqrt{\ln t}\right)^{-\frac{d}{2}}$. Cette dispersion est plus rapide, étant altérée par un facteur logarithmique, ce qui est très clairement un effet de la non-linéarité, et surtout du fait que le potentiel induit par cette non-linéarité a un puits infini lorsque la solution tend vers 0. De plus, le module de la solution rééchelonnée tend vers un profil gaussien universel. La preuve repose sur le fait que la densité $\rho(t) = |v(t)|^2$ vérifie une équation de Fokker-Planck avec potentiel quadratique au premier ordre, avec des termes sources négligeables lorsque $t \to \infty$. En outre, il est également montré par la suite que les normes de Sobolev d'indice positif pour u croissent logarithmiquement lorsque $t \to \infty$.

Régime défocalisant et limite semi-classique. Une autre question concerne l'analyse semi-classique de logNLS, à savoir l'étude de la solution u^ε de

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon + \lambda u^\varepsilon \ln|u^\varepsilon|^2 = 0. \tag{3.6}$$

A notre connaissance, la seule étude d'une analyse semi-classique de cette équation se trouve dans [36]. Dans cet article, une analyse BKW de logNLS dans le cas défocalisant ($\lambda < 0$) est effectuée, permettant de se ramener à l'étude du système

$$\left\{\begin{array}{l} \partial_t v^\varepsilon + (v^\varepsilon \cdot \nabla)v^\varepsilon - \lambda\nabla\left(\ln|a^\varepsilon|^2\right) = 0, \\ \partial_t a^\varepsilon + v^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2}a^\varepsilon \operatorname{div} v^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon, \end{array}\right. \tag{3.7}$$

avec la relation (similaire aux autres non-linéarités) :

$$\phi^\varepsilon(t,x) = \phi_{\text{in}}(x) - \int_0^t \left(\frac{1}{2}|v^\varepsilon(\tau,x)|^2 + \lambda\ln|a^\varepsilon(\tau,x)|^2\right) \mathrm{d}\tau.$$

La solution de logNLS s'écrit alors $u^\varepsilon = a^\varepsilon e^{i\frac{\phi^\varepsilon}{\varepsilon}}$ comme précédemment. A partir de celle-ci sont construites des solutions en dimension $d = 1$ dans des espaces de Zhidkov

$$X^s(\mathbb{R}) = \left\{f \in L^\infty(\mathbb{R}), f' \in H^{s-1}(\mathbb{R})\right\}.$$

Pour $s \geq 1$, cet espace est une algèbre, contenant en particulier l'espace de Sobolev $H^s(\mathbb{R})$. Cependant, les données initiales considérées sont supposées être non seulement dans $X^s(\mathbb{R})$ pour $s > \frac{5}{2}$, mais surtout loin du vide, à savoir plus précisément

$$|u^\varepsilon(0,x)| = \sqrt{\rho_0(x)} \geq \rho_{0*}, \qquad \forall x \in \mathbb{R}, \forall\varepsilon > 0,$$

pour une constante $\rho_{0*} > 0$. Cette hypothèse est faite afin d'éviter la singularité du logarithme, ce qui permet de se retrouver en un certain sens avec une non-linéarité lisse. Les données initiales considérées ne sont donc ni L^2 ni H^s. A partir de ces hypothèses, les auteurs prouvent alors la convergence de $(a^\varepsilon, v^\varepsilon)$ vers (a, v) solution de

$$\left\{\begin{array}{l} \partial_t v + (v \cdot \nabla)v - \lambda\nabla\left(\ln|a|^2\right) = 0, \\ \partial_t a + v \cdot \nabla a + \frac{1}{2}a \operatorname{div} v = 0. \end{array}\right. \tag{3.8}$$

Ce système est la version symétrisée du *système d'Euler isotherme*

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) - \lambda \nabla \rho = 0, \end{cases} \tag{3.9}$$

avec $\rho = |a|^2$ (voir [45, 113]), dont le problème de Riemann a été résolu par B. Riemann en dimension $d = 1$ dans son mémoire à la Royal Academy of Sciences of Göttinger (1860) en introduisant les invariants de Riemann $v \pm \sqrt{2\lambda} \ln \rho$ (voir [133, p.126]). Ce système a par ailleurs été étudié dans de nombreux contextes (par exemple [143, 22, 62, 47, 11]).

Dans le cadre précédent, ces considérations permettent d'obtenir par la suite la convergence de la transformée de Wigner vers une mesure (de Wigner) monocinétique $\mu(t, \mathrm{d}x, \mathrm{d}\xi) = \rho(t, x)\, \mathrm{d}x \otimes \delta_{\xi = v(t,x)}$. Cette mesure est alors solution de l'équation cinétique

$$\partial_t f + \xi \cdot \nabla_x f + \lambda \nabla_x (\ln \rho) \cdot \nabla_\xi f = 0. \tag{3.10}$$

Cette équation, appelée *système d'Euler isotherme cinétique*, a également été étudiée, en particulier comme *limite quasi-neutre* du système de Vlasov-Poisson avec électrons sans masse ([93, 80]), ou comme généralisation du système d'Euler isotherme puisque toute mesure monocinétique solution de celle-ci, de la forme $\rho(t, x)\, \mathrm{d}x \otimes \delta_{\xi = v(t,x)}$, doit alors être telle que (ρ, v) soit solution du système d'Euler isotherme.

Cependant, ces études se font toutes pour des solutions loin du vide, évitant le problème de la singularité du logarithme. Complémentairement, R. Carles et A. Nouri ([36]) ont également traité le cas de données gaussiennes, montrant que les considérations précédentes restaient vraies dans ce cadre.

Enfin, on notera que les résultats de dynamique universelle pour logNLS dans le cas défocalisant ont également été étendus au système d'Euler isotherme par R. Carles, M. Hillairet et K. Carrapatoso [30], et même au système généralisé avec une loi de pression convexe P vérifiant $P'(0) > 0$ ainsi qu'un terme de Korteweg et un terme de Navier-Stokes quantique :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla P(\rho) = \dfrac{\delta^2}{2} \rho \nabla \Big(\dfrac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \Big) + \nu \operatorname{div}(\rho D v), \end{cases}$$

avec $\delta, \nu \geq 0$.

Cas focalisant : le Gausson. Au contraire, lorsque $\lambda > 0$, le comportement de r change complètement : elle devient alors périodique. Ceci indique que l'équation (3.1) est alors dans un régime focalisant. Cette conjecture a été rapidement prouvée par T. Cazenave en étudiant l'énergie comme une fonctionnelle, prouvant le résultat suivant :

Lemme 3.2 ([38])**.** *Pour tout $k < \infty$ tel que*

$$L_k \coloneqq \big\{ v \in W(\mathbb{R}^d), \|v\|_{L^2} = 1, \mathcal{E}(v) \leq k \big\} \neq \emptyset,$$

on a

$$\inf_{\substack{v \in L_k \\ 1 \leq p \leq \infty}} \|v\|_{L^p} > 0.$$

En particulier, ce résultat permet de prouver que n'importe quelle solution de (3.1) voit ses normes L^p (pour $p \in [1, \infty]$) être minorées par une constante strictement positive, montrant rigoureusement le caractère non-dispersif de logNLS dans le cas $\lambda > 0$.

Dans ce cadre, il est alors naturel de considérer le problème de l'existence de solitons, tout comme pour le cas d'une non-linéarité de type puissance, ce qui revient à trouver les solutions (réelles) dans H^1 de l'équation

$$\frac{1}{2} \Delta \psi_\omega + \lambda \ln(|\psi_\omega|^2) \psi_\omega = \omega\, \psi_\omega, \tag{3.11}$$

pour ω réel. De nouveau, le manque de régularité du logarithme pose problème pour une telle équation de Schrödinger stationnaire. Cependant, une première propriété de cette équation permet de réduire le problème : dans la continuité de l'invariance d'échelle pour (3.1), si ψ_ω est solution de (3.11), alors $\psi_\omega e^{-\frac{\omega}{2\lambda}}$ est solution de (3.11) pour $\omega = 0$. Dès lors, nous avons besoin de ne considérer que l'équation

$$\frac{1}{2} \Delta \psi_\omega + \lambda \ln(|\psi_\omega|^2) \psi_\omega = 0,$$

et d'étudier ses solutions dans H^1. Pour cette équation, on peut citer le résultat [58] qui montre l'existence d'une infinité de solutions radiales dans $H^1(\mathbb{R}^d)$. Cependant, dans le même article, il est également montré que, parmi toutes les solutions, la seule solution $\mathcal{C}^2$ strictement positive est (aux translations près) la fonction suivante :

$$G^d(x) \coloneqq \exp\Big[\frac{d}{2} - \lambda |x|^2\Big]. \tag{3.12}$$

Cette fonction (déjà remarquée dans [23]) correspond en fait au cas où $A(t)$ est constante égale à λI_d dans l'exemple précédent de données gaussiennes. Plus communément appelée *Gausson*, sa stabilité (orbitale) pour (3.1) a déjà été étudiée puis prouvée dans [8], dans la suite du travail de Cazenave [38] pour le cas radial, en utilisant des résultats fins liés aux inégalités de Sobolev logarithmiques.

On notera par ailleurs que les solitons existent donc pour n'importe quel $\omega \in \mathbb{R}$, contrairement au cas d'une non-linéarité de type puissance ou lipschitzienne où ils n'existent que pour $\omega > 0$. De manière plus générale, les solutions gaussiennes générales voient leur variance rester bornée et minorée par une constante strictement positive au cours du temps, qu'elles soient périodiques (en dimension $d = 1$ par exemple) ou non. Ainsi, ces solutions forment une famille de strutures non-dispersives ne se décomposant pas comme une somme de Gaussons (plus un terme dispersif) en temps long, alors même qu'il ne s'agit pas de solutions à l'équation de Schrödinger stationnaire (3.11). Ceci est une propriété inhabituelle pour une équation de type NLS : ce genre de comportements n'a jamais été observé pour des non-linéarités de type puissance.

Résultats numériques. La non-linéarité logarithmique, et en particulier sa singularité en 0, est également très problématique dans le cadre numérique, où les hypothèses habituelles demandent à la non-linéarité d'être assez régulière. Cette problématique a intéressé W. Bao, R. Carles, C. Su et Q. Tang, qui ont étudié la convergence de divers schémas numériques approximant la solution. Une première approche apportée par ces auteurs ([12, 13]) consiste en une régularisation du logarithme, en remplaçant $\ln(|u(t)|^2)$ par $\ln(\varepsilon + |u(t)|^2)$ pour une valeur $\varepsilon > 0$ assez petite. De nombreuses simulations numériques accompagnent cette étude et révèlent plusieurs autres phénomènes nouveaux dans le régime focalisant concernant l'interaction entre Gaussons ou même entre gaussiennes générales.

En particulier, les interactions entre 2 Gaussons sans vitesse donnent numériquement des comportements substantiellement différents selon la distance entre les 2 Gaussons : lorsque ceux-ci sont assez éloignés, la structure semble se conserver relativement longtemps ; lorsque les 2 Gaussons sont plus proches, ceux-ci sont attirés l'un l'autre et se croisent, avant de recommencer encore et encore, s'éloignant de moins en moins entre chaque croisement, tandis que plusieurs petites structures allant à l'infini apparaissent. Enfin, d'autres simulations semblent également indiquer non seulement l'existence de multi-Gaussons et même de multi-gaussiennes (solution se comportant en temps grand comme une somme de solutions gaussiennes s'éloignant les unes des autres), mais également leur stabilité dès lors que la distance entre chaque gaussienne reste assez grande ou, plus probablement, avec une vitesse relative non-nulle.

On notera également que les mêmes auteurs ont proposé récemment une autre approche, en régularisant l'énergie à la place de la non-linéarité directement ([14]).

4. Problèmes mathématiques

Comparé aux équations de Schrödinger avec non-linéarité de type puissance, l'étude mathématique de logNLS est relativement peu poussée. Dès lors, de nombreuses problématiques peuvent se poser, la plupart étant des parallèles des résultats connus pour (2.1). On énumère dans cette section certains de ces problèmes, que nous nous sommes posés durant cette thèse.

4.1. Analyse semi-classique

Exception faite du cas gaussien, non seulement l'analyse semi-classique de (3.6) n'a pour le moment été faite que dans un cadre loin du vide, mais l'analyse des équations limites n'a également été effectuée que dans ce même cadre. Il semble donc pertinent de s'intéresser à ces problématiques avec le cadre des théories de Cauchy dans les espaces de Sobolev décris précédemment pour logNLS.

Par ailleurs, seul le régime défocalisant a été étudié dans ce cadre semi-classique, il serait donc intéressant de l'étudier dans le cas focalisant. Cependant, les équations limites semblent peu adaptées au cadre Sobolev lorsque $\lambda > 0$. En effet, si on reprend l'équation (3.9), en notant l'inconnue de Riemann $\psi = \ln \rho$, (v, ψ) satisfait le système :

$$\begin{cases} \partial_t v + (v \cdot \nabla)v + \lambda \nabla\psi = 0, \\ \partial_t \psi + v \cdot \nabla\psi + \operatorname{div} v = 0. \end{cases} \tag{4.1}$$

En négligeant les termes de convection non-linéaire, ψ vérifierait alors

$$\partial_{tt}^2 \psi + \lambda \Delta \psi = 0,$$

qui est une équation elliptique singulière n'ayant de solution que dans le cadre des fonctions analytiques, dont la transformée de Fourier est, moralement, exponentiellement décroissante (voir par exemple [120] en dimension $d = 1$ et [107] en dimension $d \geq 2$). Dès lors, ψ devrait tendre vers 0 à l'infini, autrement dit ρ devrait tendre vers 1 à l'infini. On peut en fait légèrement améliorer ces hypothèses, mais il semble cependant plus approprié de regarder (3.6) lorsque $\varepsilon \to 0$ dans un cadre peu éloigné de Gross-Pitaevskii.

4.2. Cas défocalisant : quantifier la convergence

Dans le cas défocalisant, la convergence de la densité $|v(t)|^2$ prouvée dans [33] est une convergence faible. Associée à la convergence des moments, il est alors connu que cette convergence est équivalente à la convergence de $|v(t)|^2$ pour la distance de Wasserstein quadratique $\mathcal{W}_2$, où les distances de Wasserstein sont définies pour des mesures de probabilité ν_1 et ν_2 par :

$$\mathcal{W}_p(\nu_1,\nu_2) = \inf\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p \,\mathrm{d}\mu(x,y)\right)^{\frac{1}{p}}; \quad (\pi_j)_\#\mu = \nu_j\right\}.$$

Dès lors, avoir une distance pour une convergence amène à la quantification de cette convergence : à quelle vitesse la distance entre $|v(t)|^2$ et γ^2 tend vers 0 ? Comme déjà expliqué précédemment, la preuve utilisée dans [33] utilise le fait que toute solution de l'équation de Fokker-Planck linéaire avec potentiel quadratique a comme limite une gaussienne. Plus précisément, les auteurs utilisent d'abord un argument de compacité pour obtenir un objet limite pour l'équation initiale, de sorte que l'équation de Fokker-Planck vérifiée par $|v(t)|^2$ peut être passée à la limite afin d'éliminer tous les termes sources négligeables.

Cependant, la vitesse à laquelle ces termes sources décroissent n'est pas utilisée dans cette preuve. La précision de cette preuve peut donc être améliorée en utilisant cette décroissance couplée aux propriétés du semi-groupe de Fokker-Planck dans le but d'une quantification de la convergence. On se tourne alors naturellement en premier lieu vers les distances de Wassertein, dont le lien avec l'équation de Fokker-Planck a déjà été étudié ([24, 50, 101, 121, 114, 145]). En particulier, la distance $\mathcal{W}_1$ est propice pour notre cadre grâce à sa représentation duale :

$$\mathcal{W}_1(\mu_1,\mu_2) = \sup\left\{\int_{\mathbb{R}^d} \Phi \,\mathrm{d}(\mu_1-\mu_2), \Phi : \mathbb{R}^d \to \mathbb{R} \text{ continue}, \mathrm{Lip}(\Phi) \leq 1\right\}.$$

Par ailleurs, il est également intéressant de confronter ce comportement en temps long avec la limite semi-classique, dès lors qu'un objet limite pour $\varepsilon \to 0$ peut être défini dans le cadre Sobolev en régime défocalisant. Cette question de la commutation entre limite semi-classique et temps long se pose déjà pour NLS, en linéaire ([156, 158, 91]) comme en non-linéaire ([28]).

4.3. Cas focalisant : expliquer les simulations numériques

Une dernière problématique concerne l'explication des simulations numériques effectuées par W. Bao, R. Carles, C. Su et Q. Tang, et expliquées précédemment dans la Section 3.2. En particulier, on peut s'attarder sur les différents problèmes suivants en régime focalisant :

— Stabilité de la structure double-Gausson sans vitesse initiale lorsque la distance entre les deux est assez grande ;
— Explication du comportement lorsque la donnée initiale est une somme de 2 Gaussons proches ;
— Stabilité des solutions Gaussiennes ;
— Existence et stabilité des multi-Gaussons et multi-gaussiennes.

La Conjecture de Résolution semble encore hors de portée, d'autant qu'il faudrait tout d'abord comprendre plus profondément la mécanique associée aux solutions gaussiennes générales, mais ces problématiques semblent de bonnes premières étapes dans ce sens.

5. Enoncé des résultats

Dans cette section, nous énonçons les résultats de cette thèse, dont les preuves seront développées dans les chapitres suivants.

5.1. Taux de convergence en distance de Wassertein et analyse semi-classique pour le cas défocalisant

Nous nous intéressons tout d'abord au régime défocalisant ($\lambda < 0$), et plus particulièrement aux deux problématiques qui nous motivaient en parallèle, à savoir la quantification de la convergence en temps long et l'analyse semi-classique. Le problème de Cauchy ayant déjà été abordé dans le cas sans constante semi-classique, son extension dans le cadre semi-classique se fait naturellement :

Théorème 5.1. *Soit $\varepsilon > 0$ et $u_{\varepsilon,\mathrm{in}} \in H^1 \cap \mathcal{F}(H^\alpha)(\mathbb{R}^d)$ pour un $0 < \alpha \leq 1$. Alors il existe une unique solution globale $u_\varepsilon \in L^\infty_{loc}\left(\mathbb{R}, H^1 \cap \mathcal{F}(H^\alpha)(\mathbb{R}^d)\right) \cap \mathcal{C}\left(\mathbb{R}, H^{-1} \cap L^2_w(\mathbb{R}^d)\right)$ de* (3.6). *De plus, sont également conservées :*

— *la masse $M(u_\varepsilon(t))$,*
— *le moment angulaire $\mathcal{J}_\varepsilon(u_\varepsilon(t)) \coloneqq \varepsilon\mathcal{J}(u_\varepsilon(t))$,*
— *l'énergie $\mathcal{E}_\varepsilon(u_\varepsilon(t)) \coloneqq \frac{\varepsilon^2}{2}\|\nabla u_\varepsilon(t)\|_{L^2} - \lambda \int_{\mathbb{R}^d} |u_\varepsilon(t,x)|^2 \ln|u_\varepsilon(t,x)|^2 \,\mathrm{d}x$.*

A partir de cette théorie de Cauchy, il est alors possible de réétablir le résultat de dynamique universelle dans le cadre semi-classique. Pour cela, de la même manière que dans [33], il est nécessaire de considérer une donnée initiale $u_{\varepsilon,\text{in}} \in H^1 \cap \mathcal{F}(H^1)(\mathbb{R}^d) \setminus \{0\}$ et d'effectuer un rescaling sur la solution $u_\varepsilon(t,x)$ de (3.6) donnée par le théorème précédent. Cependant, ce rescaling, bien que similaire à celui utilisé par R. Carles et I. Gallagher, se fait ici dans le cadre semi-classique et doit donc rendre compte des fortes oscillations, que l'on fait apparaître dans l'exponentielle complexe de (3.5) :

$$u_\varepsilon(t,x) = \frac{1}{\tau(t)^{\frac{d}{2}}} \frac{\|u_{\text{in}}\|_{L^2}}{\|\gamma\|_{L^2}} v_\varepsilon\left(t, \frac{x}{\tau(t)}\right) e^{i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2\varepsilon}}, \tag{5.1}$$

avec un rescaling similaire pour la donnée initiale. La conservation de l'énergie, combinée à ce rescaling, permet d'obtenir une borne uniforme en temps pour v_ε dans $H^1 \cap \mathcal{F}(H^\alpha)(\mathbb{R}^d)$ et même plus précisément, en définissant

$$\tilde{E}_\varepsilon(t,v) = \frac{\varepsilon^2}{\tau(t)^2}\|\nabla_y v(t)\|^2_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d}\left(|y|^2 + \left|\ln|v(t,y)|^2\right|\right)|v(t,y)|^2\,\mathrm{d}y,$$

dès lors que $\tilde{E}_\varepsilon(0, v_{\varepsilon,\text{in}})$ est uniformément borné en ε (ce qui est par exemple le cas pour des données initiales de type BKW sous de bonnes hypothèses), alors $\tilde{E}_\varepsilon(t,v)$ est uniformément borné en $t \geq 0$ et $\varepsilon > 0$.

A partir de là, l'idée générale est d'utiliser une approche hydrodynamique. Formellement, si $v_\varepsilon(t,y) = a_\varepsilon(t,y)e^{i\frac{\phi_\varepsilon(t,y)}{\varepsilon}}$ avec a_ε et ϕ_ε à valeurs réelles, alors on obtient le système d'équations

$$\begin{cases} \partial_t\phi_\varepsilon + \dfrac{1}{2\tau(t)^2}\nabla\phi_\varepsilon\cdot\nabla\phi_\varepsilon - \lambda\ln\left|\dfrac{a_\varepsilon}{\gamma}\right|^2 = \dfrac{\varepsilon^2}{2\tau(t)^2}\dfrac{\Delta a_\varepsilon}{a_\varepsilon}, \\ \partial_t a_\varepsilon + \dfrac{1}{\tau(t)^2}\nabla\phi_\varepsilon\cdot\nabla a_\varepsilon + \dfrac{1}{2\tau(t)^2}a_\varepsilon\Delta\phi_\varepsilon = 0. \end{cases}$$

De là, en notant $V_\varepsilon = \nabla\phi_\varepsilon$, les observables quadratiques a_ε^2 et $a_\varepsilon V_\varepsilon$ vérifient une équation d'Euler avec termes sources

$$\begin{cases} \partial_t a_\varepsilon^2 + \dfrac{1}{\tau(t)^2}\operatorname{div}(a_\varepsilon V_\varepsilon) = 0, \\ \partial_t(a_\varepsilon V_\varepsilon) + \dfrac{1}{\tau(t)^2}\operatorname{div}(a_\varepsilon V_\varepsilon\otimes V_\varepsilon) - \lambda\nabla a_\varepsilon^2 + 2\lambda y a_\varepsilon^2 = \dfrac{\varepsilon^2}{4\tau(t)^2}\nabla\Delta a_\varepsilon^2. \end{cases}$$

Si ϕ_ε peut ne pas être bien définie, les observables quadratiques quant à elles le sont rigoureusement à partir de v_ε via les formules

$$\rho_\varepsilon := |v_\varepsilon|^2, \qquad J_\varepsilon := \operatorname{Im}(\varepsilon\overline{v_\varepsilon}\nabla v_\varepsilon).$$

Avec ces définitions, le système précédent devient alors

$$\begin{cases} \partial_t\rho_\varepsilon + \dfrac{1}{\tau(t)^2}\operatorname{div}(a_\varepsilon v_\varepsilon) = 0, \\ \partial_t J_\varepsilon - \lambda\nabla\rho_\varepsilon + 2\lambda y\rho_\varepsilon = \dfrac{\varepsilon^2}{4\tau(t)^2}\nabla\Delta\rho_\varepsilon - \dfrac{\varepsilon^2}{\tau^2(t)}\nabla\cdot(\operatorname{Re}(\nabla v_\varepsilon\otimes\overline{\nabla v_\varepsilon})). \end{cases}$$

Via le changement de variable en temps $s = \frac{1}{2}\ln\dot\tau(t)$ et en introduisant $\nu_\varepsilon := \frac{\varepsilon^2}{\tau^2(t)}\operatorname{Re}\left(\nabla v_\varepsilon\otimes\overline{\nabla v_\varepsilon}\right)$, on obtient que ρ_ε satisfait

$$\partial_s\rho_\varepsilon - L\rho_\varepsilon = \frac{2\lambda}{\dot\tau^2}\partial_s\rho_\varepsilon - \frac{\lambda}{\dot\tau^2}\partial_s^2\rho_\varepsilon - \frac{\varepsilon^2}{4\,\lambda\,\tau^2}\Delta^2\rho_\varepsilon + \frac{1}{\lambda}\nabla\cdot(\nabla\cdot\nu_\varepsilon),$$

où $L = \Delta + \nabla\cdot(2y\,.)$ est l'opérateur de Fokker-Planck quadratique. On remarque alors que les termes principaux sont dans le membre de gauche, tous les termes de droite étant négligeables (en un certain sens). En utilisant les propriétés de l'opérateur de Fokker-Planck défini précédemment et du semi-groupe induit par celui-ci, nous démontrons non seulement la convergence en distance de Wasserstein $\mathcal{W}_1$ de ρ_ε vers γ^2, mais également une vitesse de convergnce explicite indépendante de $\varepsilon > 0$ (sous réserve des hypothèses de type BKW sur la donnée initiale). Certains moments ont par ailleurs une évolution explicite et sont également donnés dans ce premier résultat :

Théorème 5.2. *Soit $\lambda > 0$, $\varepsilon > 0$ et $u_{\varepsilon,\text{in}} \in H^1 \cap \mathcal{F}(H^1)(\mathbb{R}^d) \setminus \{0\}$. On définit $v_\varepsilon = v_\varepsilon(t,y)$ à partir de u_ε donnée par le Théorème 5.1 via la relation* (5.1). *Il existe une fonction continue croissante $C : [0,\infty) \to [0,\infty)$ ne dépendant que de λ et d telle que pour tout $t \geq 2$,*

$$\int_{\mathbb{R}^d} y\,|v_\varepsilon(t,y)|^2\,\mathrm{d}y = \frac{1}{\tau(t)}\frac{\|\gamma^2\|_{L^1}}{\|u_{\varepsilon,\text{in}}\|^2_{L^2}}(I^\varepsilon_{1,0}\,t + I^\varepsilon_{2,0}) \underset{t\to\infty}{\longrightarrow} 0,$$

$$\left|\int_{\mathbb{R}^d} |y|^2 \, |v_\varepsilon(t,y)|^2 \, \mathrm{d}y - \int_{\mathbb{R}^d} |y|^2 \, \gamma^2(y) \, \mathrm{d}y\right| \leq C\Big(\tilde{E}_\varepsilon(0, v_{\varepsilon,in})\Big) \, \frac{\dot{\tau}(t)+1}{\dot{\tau}(t)^2} \underset{t\to\infty}{\longrightarrow} 0,$$
$$\mathcal{W}_1\left(\frac{|v_\varepsilon(t,.)|^2}{\pi^{\frac{d}{2}}}, \frac{\gamma^2}{\pi^{\frac{d}{2}}}\right) \leq C\Big(\tilde{E}_\varepsilon(0, v_{\varepsilon,in})\Big) \, \frac{1}{\sqrt{\ln t}}.$$

où

$$I_{1,0}^\varepsilon = \operatorname{Im} \varepsilon \int_{\mathbb{R}^d} \overline{u_{\varepsilon,\mathrm{in}}} \, \nabla u_{\varepsilon,\mathrm{in}} \, \mathrm{d}y, \qquad I_{2,0}^\varepsilon = \int_{\mathbb{R}^d} y \, |u_{\varepsilon,\mathrm{in}}|^2 \, \mathrm{d}y.$$

Grâce au fait que $\tilde{E}_\varepsilon(t, v_\varepsilon)$ est uniformément bornée en $t \geq 0$ et $\varepsilon > 0$ dès lors que $\tilde{E}_\varepsilon(0, v_{\varepsilon,in})$ l'est, la transformée de Wigner de v_ε a de bonnes propriétés pour pouvoir passer à la limite semi-classique $\varepsilon \to 0$. Le résultat précédent, indépendant de $\varepsilon > 0$, donne alors une caractérisation du comportement en temps long de la mesure de Wigner limite qui en résulte. Ces propriétés se reportent par ailleurs sur la transformée de Wigner de u_ε directement, ce qui permet d'obtenir les propriétés suivantes, en dénotant $\mathcal{M}(\mathbb{R}^d)$ l'ensemble des mesures finies positives sur $\mathbb{R}^d$, $\mathcal{P}(\mathbb{R}^d)$ l'ensemble des mesures de probabilité et

$$L^1_2(\mathbb{R}^d) := \left\{ f \in L^1(\mathbb{R}^d), \int_{\mathbb{R}^d} |y|^2 \, |f(y)| \, \mathrm{d}y < \infty \right\},$$
$$L \log L(\mathbb{R}^d) := \left\{ f \in L^1(\mathbb{R}^d), |f| \log |f| \in L^1(\mathbb{R}^d) \right\},$$
$$\mathcal{P}_j(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d), \int |x|^j \, \mathrm{d}\mu < \infty \right\} \qquad \text{doté de la distance } \mathcal{W}_j \text{ pour } j = 1, 2.$$

Théorème 5.3. *Pour toute famille $u_{\varepsilon,\mathrm{in}} \in H^1 \cap \mathcal{F}(H^1)(\mathbb{R}^d) \setminus \{0\}$ ($\varepsilon > 0$) tel que $\tilde{E}_\varepsilon(0, u_{\varepsilon,\mathrm{in}})$ est uniformément borné en $\varepsilon > 0$, on définit, pour tout $\varepsilon > 0$, u_ε fourni par le Théorème 5.1, v_ε par (5.1), et W_ε (resp. $\tilde{W}_\varepsilon$) la transformée de Wigner de u_ε (resp. v_ε). Alors il existe une sous-suite $(\varepsilon_n)_n$ tel que $\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$ et deux mesures finies (positives) W et $\tilde{W}$ in $L^\infty((0,\infty), \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d))$ tel que pour tout $p \in [1, \infty)$*

$$W_{\varepsilon_n} \underset{n\to\infty}{\rightharpoonup} W \qquad \textit{dans } L^p_{loc}((0,\infty), \mathcal{A}'), \qquad\qquad \tilde{W}_{\varepsilon_n} \underset{n\to\infty}{\rightharpoonup} \tilde{W} \qquad \textit{dans } L^p_{loc}((0,\infty), \mathcal{A}'),$$

et la relation entre W_ε et $\tilde{W}_\varepsilon$

$$W_\varepsilon(t,x,\xi) = \frac{\|u_{\varepsilon,\mathrm{in}}\|^2_{L^2}}{\|\gamma^2\|_{L^1}} \, \tilde{W}_\varepsilon\left(t, \frac{x}{\tau(t)}, \tau(t)\,\xi - \dot{\tau}(t)\,x\right)$$

passe à la limite par le fait que $\|u_{\varepsilon_n,\mathrm{in}}\|_{L^2}$ converge (vers une limite notée $M_0 \geq 0$) quand $n \to \infty$. De plus, on a

$$\pi^{-\frac{d}{2}} \, \tilde{\rho}(t,y) := \pi^{-\frac{d}{2}} \int_{\mathbb{R}^d} \tilde{W}(t,y,\mathrm{d}\eta) \in L^\infty((0,\infty), L^1_2 \cap L\log L(\mathbb{R}^d)) \cap \mathcal{C}(\mathbb{R}^+, \mathcal{P}_1(\mathbb{R}^d)),$$
$$\tilde{\rho}(t, \mathbb{R}^d) = \|\gamma^2\|_{L^1} \qquad \textit{pour tout } t \geq 0,$$

et il existe $C_0 > 0$ tel que

$$\sup_{t\geq 0} \operatorname{ess} \frac{1}{\tau(t)^2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\eta|^2 \, \tilde{W}(t,\mathrm{d}y,\mathrm{d}\eta) + \int_0^\infty \frac{\dot{\tau}(t)}{\tau^3(t)} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\eta|^2 \, \tilde{W}(t,\mathrm{d}y,\mathrm{d}\eta) \, \mathrm{d}t \leq C_0,$$
$$\int_{\mathbb{R}^d} y \, \tilde{\rho}(t,y) \, \mathrm{d}y = \frac{1}{\tau(t)}(C_1 t + C_2), \qquad \forall t \geq 0,$$

où

$$C_j = \lim_{n\to\infty} \frac{\|\gamma^2\|_{L^1}}{\|u_{\varepsilon_n,\mathrm{in}}\|^2_{L^2}} \, I^{\varepsilon_n}_{j,0} \qquad \textit{pour } j = 1, 2,$$

ce qui implique que

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \end{pmatrix} \tilde{\rho}(t,y) \, \mathrm{d}y \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \end{pmatrix} \gamma^2(y) \, \mathrm{d}y.$$

Enfin, il existe $C_3 > 0$ tel que pour tout $t \geq 2$,

$$\mathcal{W}_1\left(\frac{\tilde{\rho}(t)}{\pi^{\frac{d}{2}}}, \frac{\gamma^2}{\pi^{\frac{d}{2}}}\right) \leq \frac{C_3}{\sqrt{\ln t}}.$$

Moralement, cette mesure de Wigner devrait être une solution de l'équation cinétique (3.10), mais encore une fois la singularité du logarithme en 0 est problématique et ne permet pas de conclure, sauf dans le cas gaussien qui est explicite (ceci a déjà été prouvé en dimension $d = 1$ dans [36]). Il est même compliqué de ne trouver ne serait-ce qu'une véritable formulation faible de (3.10) bien posée pour des mesures finies sur l'espace $\mathbb{R}^d$ entier. Pour le moment, à notre connaissance, aucune théorie de Cauchy n'a d'ailleurs été trouvée dans ce cadre. Pour autant, les arguments précédents pourraient être réemployés et adaptés à cette équation (si tant est qu'une théorie de Cauchy existe), tout du moins formellement. En particulier, les quantités (formellement) conservées et équations (formelles) sur les moments permettraient d'obtenir le même comportement pour une solution globale que pour la mesure de Wigner que nous avons trouvée.

Pour aller plus loin, nous avons vu des solutions gaussiennes explicites pour (3.1) (et donc également pour (3.6)), dont le parallèle pour (3.9) sont des solutions explicites où ρ est une gaussienne et v un polynôme du second degré, et pour (3.10) la mesure monocinétique correspondante. Nous pouvons également considérer des solutions sous forme de mesures gaussiennes pour (3.10), donnant de nouvelles solutions explicites dans un cadre différent des études précédemment citées pour cette équation. En particulier, les considérations utilisées dans la discussion précédente sont alors rigoureuses pour de telles solutions globales. Leur comportement à l'infini est même plus précis, tout comme pour les solutions gaussiennes de (3.1), puisque la convergence (après rescaling) se fait en norme L^1.

Théorème 5.4. *1. Pour $c_{1,0} > 0$, $c_{2,0} > 0$ et $c_{1,1}, B_0, B_1 \in \mathbb{R}$, on définit $c_1 \in \mathcal{C}^\infty(\mathbb{R}^+)$ solution de*

$$\ddot{c}_1 = \frac{2\lambda}{c_1} + \frac{\tilde{C}^2}{c_1^3}, \qquad \tilde{C} := c_{1,0}\, c_{2,0}, \qquad c_1(0) = c_{1,0}, \qquad \dot{c}_1(0) = c_{1,1}. \tag{5.2}$$

On définit également

$$c_2(t) := \frac{\tilde{C}}{c_1(t)}, \qquad b_1(t) := B_1 t + B_0, \qquad b_2(t,x) := \frac{\dot{c}_1(t)}{c_1(t)}(x - B_1\, t - B_0) + B_1. \tag{5.3}$$

Alors la fonction $f = f(t,x,\xi)$ définie par

$$f(t,x,\xi) = \frac{1}{\pi\, c_1(t)\, c_2(t)} \, \exp\left[-\frac{|x-b_1(t)|^2}{c_1(t)^2} - \frac{|\xi - b_2(t,x)|^2}{c_2(t)^2}\right]$$

est solution de (3.10). *De plus, en définissant $\tilde{f} = \tilde{f}(t,y,\eta)$ via*

$$f(t,x,\xi) := \frac{1}{\|\gamma^2\|_{L^1}}\, \tilde{f}\left(t, \frac{x}{\tau(t)}, \tau(t)\,\xi - \dot{\tau}(t)\, x\right),$$

et

$$\tilde{\rho}(t,y) := \int_{\mathbb{R}^d} \tilde{f}(t,y,\eta)\,\mathrm{d}\eta,$$

on a alors

$$\|\tilde{\rho}(t,.) - \gamma^2\|_{L^1} = \mathrm{O}\left(\sqrt{\frac{\ln\ln t}{\ln t}}\right).$$

2. *Soit $T \in (0,+\infty]$, $b_1 = b_1(t) \in \mathcal{C}^1([0,T),\mathbb{R})$, $c_1 = c_1(t) \in \mathcal{C}^1([0,T),(0,\infty))$, $b_2 = b_2(t,x) \in \mathcal{C}^1([0,T)\times\mathbb{R},\mathbb{R})$ et $c_2 = c_2(t,x) \in \mathcal{C}^1([0,T)\times\mathbb{R},(0,\infty))$ tels que*

$$f(t,x,\xi) = \frac{1}{\pi\, c_1(t)\, c_2(t,x)} \, \exp\left[-\frac{|x-b_1(t)|^2}{c_1(t)^2} - \frac{|\xi - b_2(t,x)|^2}{c_2(t,x)^2}\right], \qquad t \in [0,T),\ x,\xi \in \mathbb{R},$$

est solution de (3.10). *Alors c_2 ne dépend pas de x, toutes les fonctions sont $\mathcal{C}^\infty$ et on a* (5.2)-(5.3).

Ce résultat fournit une large famille de solutions explicites de l'équation (3.10). Un intérêt peut être trouvé pour ces solutions en analyse numérique par exemple, puisqu'elles fournissent de nouvelles solutions de référence.

5.2. Données gaussiennes et superposition non-linéaire

Nous nous plaçons ici en régime focalisant ($\lambda > 0$).

Analyse des breathers et principe de superposition non-linéaire. Nous nous intéressons tout d'abord aux simulations de [13]. En particulier, nous commençons par une première explication pour le cas de la somme de deux Gaussons éloignés sans vitesse relative. Pour cela, une approche "naïve" consiste à quantifier la "perte" de non-linéarité en considérant la somme de deux Gaussons (notés disons G_1 et G_2). Plus précisément, on essaie de quantifier la différence entre le terme non-linéaire de la somme, à savoir pour notre équation $\lambda(G_1+G_2)\ln|G_1+G_2|^2$, et la somme des non-linéarités, soit $\lambda G_1 \ln|G_1|^2 + \lambda G_2 \ln|G_2|^2$. Dans le cas d'une non-linéarité de type puissance ou lisse, cela revient souvent à effectuer une linéarisation des perturbations autour d'un soliton. Cependant, linéariser un logarithme, en particulier autour d'un Gausson dont les valeurs tendent (très rapidement) vers 0 à l'infini, ne peut se faire que formellement : une linéarisation de $\ln|G+\delta| = \ln|G| + \ln\left|1+\frac{\delta}{G}\right|$ n'est valable rigoureusement que si δ est petit par rapport à G, ce qui n'est *a priori* pas le contexte espéré. Ce problème de linéarisation pour une non-linéarité logarithmique a par ailleurs déjà été observé pour l'équation de Korteweg-de Vries logarithmique (logKdV) ([37, 127]).

Pour autant, la différence précédente peut tout de même être estimée, et ce pour n'importe quelle fonction gaussienne. Ainsi, cette approche permet d'obtenir un résultat non seulement pour une somme de Gaussons, mais même plus généralement pour une somme de solutions gaussiennes générales. Pour cela, nous commençons par revoir les caractéristiques générales des solutions gaussiennes, et en particulier de celles périodiques (comme en dimension $d=1$) appelées *breathers*. Par la suite, nous obtenons le résultat suivant, montrant que la solution de (3.1), avec comme donnée initiale la somme de plusieurs gaussiennes éloignées les unes des autres de l'ordre de $\frac{1}{\varepsilon}$, reste proche de la somme de ces Gaussons jusqu'à un temps de l'ordre de $\frac{1}{\varepsilon^2}$. Pour cela, on notera $G^d_{A_{\text{in}},\omega,x_0,v,\theta}$ la solution gaussienne ayant pour donnée initiale

$$G^d_{A_{\text{in}},\omega,x_0,v,\theta}(0,x) := \exp\left[i\,(\theta - v\cdot x) + \omega + \frac{d}{2} - x^\top A_{\text{in}} x\right].$$

Théorème 5.5. *Il existe $C_d > 0$ (ne dépendant que de la dimension d) telle que la propriété suivante est vraie. Soient $\lambda > 0$, $N \in \mathbb{N}^*$, et considérons $x_k \in \mathbb{R}^d$, A^k_{in} matrices complexes symétriques dont la partie réelle est définie-positive, $\omega_k \in \mathbb{R}$ et $\theta_k \in \mathbb{R}$ pour $k = 1,\dots,N$ et $v \in \mathbb{R}^d$. Soit u la solution de* (3.1) *avec donnée initiale $u_{\text{in}}(x) := \sum G^d_{A^k_{\text{in}},\omega_k,x_k,v,\theta_k}(0,x)$ pour tout $x \in \mathbb{R}^d$, et $G(t) := \sum G^d_{A^k_{\text{in}},\omega_k,x_k,v,\theta_k}(t)$. On note également $A_k(t)$ la solution de* (3.4) *avec donnée initialle A^k_{in} et*

$$\tau_- := \inf_{t,k,j} \sigma(\operatorname{Re} A_k(t)), \qquad \tau_+ := \sup_{t,k,j} \sigma(\operatorname{Re} A_k(t)),$$

où $\sigma(M)$ désigne le spectre d'une matrice M.

Alors $0 < \tau_- \leq \tau_+ < \infty$ et il existe $\varepsilon_0 > 0$ ne dépendant que de $\delta\omega := \max_k |\omega_k - \omega_{k+1}|$, τ_-, τ_+ et N tel que, si

$$\varepsilon := \left(\min_k |x_{k+1} - x_k|\right)^{-1} < \varepsilon_0,$$

alors pour tout $t \geq 0$,

$$\|u(t) - G(t)\|_{L^2(\mathbb{R}^d)} \leq C_d N^{\frac{3}{2}} \frac{\lambda\,\tau_+}{\varepsilon^{\frac{d}{2}+1}\sqrt{\tau_-}} \exp\left[-\frac{\tau_-}{4\varepsilon^2} + \max_j \omega_j + 2\lambda t\right].$$

Existence de multi-Gaussons et multi-gaussiennes. Au vu des des simulations numériques ([13]) et des breathers qui sont des solutions périodiques n'étant pas des points critiques pour l'énergie, il semble que l'équation (3.1) ne soit pas complètement intégrable. De plus, les techniques d'énergie pour l'existence de multi-solitons pour (2.1) demandent une non-linéarité $\mathcal{C}^1([0,+\infty[)$. Plus précisément, ces techniques utilisent une linéarisation de l'énergie autour d'un soliton, de sorte que la hessienne de l'énergie est utilisée comme fonctionnelle pour contrôler la différence entre la solution et une somme de solitons de paramètres bien choisis. Le logarithme étant singulier en 0 et la linéarisation n'étant que formelle (comme déjà discuté précédemment), ces résultats ne permettent pas de conclure en l'état. Cependant, le calcul formel de la linéarisation nous donne une "hessienne" (formelle) autour du Gausson (3.12) extrêmement intéressante, à savoir un opérateur de Schrödinger avec potentiel quadratique :

$$H = -\frac{1}{2}\Delta + 2\lambda^2|x|^2 - \lambda d.$$

Cet opérateur aurait également toutes les propriétés nécessaires pour appliquer les résultats d'existence et de stabilité de multi-solitons, à savoir que les valeurs propres négatives de cet opérateur ont comme vecteurs propres associés respectifs G^d et $\nabla G^d = 2\lambda x G^d$. Le potentiel quadratique dans cet opérateur, tendant vers $+\infty$ à l'infini, a par ailleurs des propriétés plus intéressantes que les potentiels issus des non-linéarités de type puissance, qui tendent vers une constante à l'infini, dans le sens où les solutions restent mieux confinées à énergie fixée (aussi grande soit-elle).

De là, notre idée est d'éviter (en un certain sens) cette singularité en "rognant" la non-linéarité lorsqu'on s'approche du vide, puisque le potentiel quadratique venant de la linéarisation est bien plus forte que nécessaire. Il est cependant indispensable de pouvoir estimer ce que l'on a "rogné". Pour cela, il est intéressant d'avoir une estimation préalable en norme L^2, ces considérations permettant par la suite d'obtenir celle en norme H^1.

Une telle perspective d'estimer en norme L^2 avant la norme H^1 a déjà été utilisée par Côte et Le Coz. Via un argument de bootstrap sur des approximations successives du multi-soliton à construire, ils montrent que la norme L^2 de ces approximations a une "meilleure" décroissance que la norme H^1 (dans le sens où on multiplie par une constante assez petite) dès lors que la vitesse relative entre les profils (possiblement des états excités) est assez grande, et cette propriété leur permet par la suite de compléter leur propriété de bootstrap. Pour des énoncés plus précis, on se référera à [53].

On peut alors regarder le résultat précédent d'un point de vue légèrement différent dans cette optique. En effet, en construisant de la même manière des approximations u_n du multi-Gausson à construire comme solution de (3.1) avec donnée initiale

$$u_n(T_n) = \sum_{k=1}^{N} G^d_{A^k_{\text{in}},\omega_k,x_k,v_k,\theta_k}(T_n, x) =: G(T_n, x),$$

pour $T_n \to \infty$ et sachant que les différents Gaussons composant cette somme s'écartent en au moins $v_* t$, on peut utiliser un résultat similaire au Théorème 5.5 à partir de T_n jusqu'à $t < T_n$. En effet, comme expliqué précédemment, on estime pour ce théorème

$$\left\| G \ln |G|^2 - \sum_{k=1}^{N} G_k \ln |G_k|^2 \right\|_{L^2},$$

où $G_k = G^d_{A^k_{\text{in}},\omega_k,x_k,v_k,\theta_k}$, en utilisant la propriété (3.3) via l'équation (3.1), ce qui permet d'obtenir plus précisément

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u - G(t)\|^2_{L^2} \leq 4\lambda \|u - G(t)\|^2_{L^2} + C e^{-\frac{\tau_-}{4\varepsilon^2}} \|u - G(t)\|_{L^2}.$$

Dans ce cadre-ci, nous avons alors $\varepsilon^{-1} \sim v_* t$ (et $\tau_- = \lambda$ pour des Gaussons), ce qui permet d'obtenir en intégrant de T_n à $t < T_n$:

$$\|u_n - G\|_{L^2} \leq C \exp\Big[-\frac{\lambda}{4} (v_* t)^2 + C_0 t \Big], \qquad \forall t < T_n.$$

Ceci permet d'obtenir une borne uniforme dans L^2 sur ces approximations u_n. Avec un argument de compacité dans L^2 (grâce à la conservation de l'énergie et à une preuve similaire à celle de [53]) et au Lemme 3.1, on obtient alors le multi-soliton voulu. Ces considérations sur la norme L^2 marchent par ailleurs également sur des gaussiennes générales à la place des Gaussons, modulo un léger changement dans la vitesse de convergence (en $e^{-\tau (v_* t)^2}$ avec $\tau < \frac{\lambda}{4}$).

De plus, dans le cas où toutes les gaussiennes considérées sont des Gaussons (et donc des états fondamentaux et des points critiques de l'énergie), on peut alors appliquer la tactique de preuve précédente pour obtenir la convergence en norme H^1. Une fois celle-ci obtenue, on complète également l'étude par une convergence en norme $\mathcal{F}(H^1)$, en appliquant la même technique que pour la norme L^2 mais en y rajoutant un poids quadratique. Le résultat complet est le suivant :

Théorème 5.6. *Soient $\lambda > 0$, $N \in \mathbb{N}^*$, $d \in \mathbb{N}^*$. Soient $(v_k)_{1 \leq k \leq N}$ et $(x_k)_{1 \leq k \leq N}$ deux familles de vecteurs dans $\mathbb{R}^d$, ainsi que $(\omega_k)_{1 \leq k \leq N}$ et $(\theta_k)_{1 \leq k \leq N}$ deux familles de nombres réels. Enfin, soit $(A^{\text{in}}_k)_{1 \leq k \leq N}$ une suite de matrices complexes symétriques de partie réelle définie positive. On définit $A_k(t)$ la solution de* (3.4) *avec données initiales A^{in}_k et*

$$v_* := \min_{j \neq k} |v_j - v_k|, \qquad G_k := G^d_{A^k_{\text{in}},\omega_k,x_k,v_k,\theta_k}, \qquad \sigma_- := \frac{1}{2} \inf_{t,k} \sigma(\operatorname{Re} A_k(t)) > 0.$$

Si $v_* > 0$, alors il existe une unique solution $u \in \mathcal{C}_b(\mathbb{R}, W(\mathbb{R}^d))$ de (3.1) *et un temps $T \in \mathbb{R}$ tels que pour tout $t \geq 0$,*

$$\left\| u(T + t) - \sum_{k=1}^{N} B_k(T + t) \right\|_{L^2} \leq e^{-\frac{\sigma_- (v_* t)^2}{4}}. \tag{5.4}$$

Lorsque toutes les matrices A_k sont égales à λI_d, alors la convergence précédente est également valable pour la norme $H^1 \cap \mathcal{F}(H^1)$.

Remarque 5.1. On notera que l'on a non seulement existence, mais également unicité pour ces multi-gaussiennes. L'unicité est cependant sujette au taux de convergence (5.4). D'autres solutions pourraient potentiellement converger vers la même somme de gaussiennes en temps long, cependant le Lemme 3.1 associé aux multi-gaussiennes construites permet de montrer que toute autre solution v de (3.1) vérifie alors :

$$\left\| v(t) - \sum_{k=1}^{N} B_k(t) \right\|_{L^2} \geq C_1 e^{-2\lambda t}, \tag{5.5}$$

pour une certaine constante $C_1 > 0$ et pour tout t assez grand. Cette remarque fait écho à l'article de Côte et Friederich [57] pour NLS.

5.3. Analyse BKW pour des données analytiques

Dans le cadre semi-classique, nous avons vu que la limite formelle de la forme hydrodynamique de logNLS était le système d'Euler isotherme (3.9). Il est par ailleurs connu que la bonne inconnue pour cette équation (lorsque $\lambda < 0$) est $\ln \rho$. Nous reportons cette remarque sur la forme hydrodynamique (3.7) de logNLS. Cependant, a^ε étant complexe, définir le logarithme de celui-ci pourrait être problématique. Pour outrepasser ce problème, nous construisons des solutions de (3.6) sous la forme

$$u^\varepsilon(t,x) = e^{\frac{\psi^\varepsilon(t,x)}{2} + i\frac{\phi^\varepsilon(t,x)}{\varepsilon}},$$

où ψ^ε et ϕ^ε satisfont le système :

$$\begin{cases} \partial_t \phi^\varepsilon + \dfrac{1}{2}\nabla\phi^\varepsilon \cdot \nabla\phi^\varepsilon - \lambda \operatorname{Re}\psi^\varepsilon = 0, \\ \partial_t \psi^\varepsilon + \nabla\phi^\varepsilon \cdot \nabla\psi^\varepsilon + \Delta\phi^\varepsilon = i\dfrac{\varepsilon}{2}\Big(\Delta\psi^\varepsilon + 2\,\nabla\psi^\varepsilon\cdot\nabla\psi^\varepsilon\Big). \end{cases} \tag{5.6}$$

Tout comme précédemment, on peut introduire $v^\varepsilon := \nabla\phi^\varepsilon$, ce qui revient à résoudre le système :

$$\begin{cases} \partial_t v^\varepsilon + (v^\varepsilon\cdot\nabla)v^\varepsilon + \lambda\,\nabla\Big(\operatorname{Re}\psi^\varepsilon\Big) = 0, \\ \partial_t \psi^\varepsilon + v^\varepsilon \cdot \nabla\psi^\varepsilon + \operatorname{div} v^\varepsilon = i\dfrac{\varepsilon}{2}\Big(\Delta\psi^\varepsilon + 2\,\nabla\psi^\varepsilon\cdot\nabla\psi^\varepsilon\Big), \end{cases} \tag{5.7}$$

avec comme relation

$$\phi^\varepsilon(t,x) = \phi^\varepsilon_{\text{in}}(x) - \int_0^t \Big(\frac{1}{2}|v^\varepsilon(\tau,x)|^2 + \lambda\operatorname{Re}\psi^\varepsilon(\tau,x)\Big)\,\mathrm{d}\tau. \tag{5.8}$$

Notons d'ores et déjà que la limite formelle de ce système est bien le système d'Euler isotherme sous sa forme "variables de Riemann" (4.1). Nous pouvons cependant aller un peu plus loin. En effet, il est intéressant de passer de (5.6) à (5.7) car les équations du système (5.6) ne font intervenir que des dérivées spatiales de ϕ^ε (sauf la dérivée temporelle). Or, c'est également le cas pour ce dernier système (5.7) avec ψ^ε : on peut donc faire de même et introduire comme inconnu $\zeta^\varepsilon := \nabla\psi^\varepsilon$ à la place de ψ^ε. On se retrouve alors avec le système

$$\begin{cases} \partial_t v^\varepsilon + (v^\varepsilon\cdot\nabla)v^\varepsilon + \lambda\,\operatorname{Re}\zeta^\varepsilon = 0, \\ \partial_t \zeta^\varepsilon + \nabla\Big(v^\varepsilon\cdot\zeta^\varepsilon\Big) + \nabla\operatorname{div} v^\varepsilon = i\dfrac{\varepsilon}{2}\Big(\nabla\operatorname{div}\zeta^\varepsilon + 2\,\nabla(\zeta^\varepsilon\cdot\zeta^\varepsilon)\Big), \end{cases} \tag{5.9}$$

et une relation similaire à (5.8) pour ψ^ε avec ζ^ε. Comme remarqué précédemment, le système limite (4.1) lorsque $\lambda > 0$ semble être singulier et devrait n'être bien défini que pour des fonctions analytiques. Nous allons donc nous positionner dans ce cadre. Plus rigoureusement, les fonctions analytiques sont définies comme étant les fonctions $f \in L^2(\mathbb{R}^d)$ dont la transformée de Fourier vérifie

$$\|f\|^2_{\mathcal{H}^\ell_\delta} =: \int_{\mathbb{R}^d} \langle\xi\rangle^{2\ell} e^{2\delta\langle\xi\rangle}\Big|\hat{f}(\xi)\Big|^2\,\mathrm{d}\xi < \infty,$$

pour $\ell, \delta > 0$. On peut alors définir l'espace des fonctions analytiques ([77]) :

$$\mathcal{H}^\ell_\delta(\mathbb{R}^d,\mathbb{R}^n) := \Big\{ f \in L^2(\mathbb{R}^d,\mathbb{R}^n), \|f\|_{\mathcal{H}^\ell_\delta} < \infty \Big\}.$$

On considérera alors des fonctions continues en temps à valeurs dans $\mathcal{H}^\ell_\delta$ où $\delta = \delta(t)$ est une fonction continue (et même $\mathcal{C}^1$ au moins) dépendant du temps, c'est-à-dire plus rigoureusement des fonctions à valeurs dans

$$\mathcal{C}(I,\mathcal{H}^\ell_\delta) := \Big\{ f \in \mathcal{C}(I,L^2), \mathcal{F}^{-1}(w_\delta \hat{f}) \in \mathcal{C}(I,\mathcal{H}^\ell_0) = \mathcal{C}(I,H^\ell) \Big\},$$

où $w_\delta := \exp\Big(\delta\langle\xi\rangle\Big)$ et I un intervalle. On introduit également les normes suivantes :

$$|||f|||_{\infty,t,\ell,\delta} := \sup_{\tau\in(0,t)} \|f(\tau)\|_{\mathcal{H}^\ell_{\delta(\tau)}},$$
$$|||f|||_{2,t,\ell,\delta} := \Big(\int_0^t \|f(\tau)\|^2_{\mathcal{H}^\ell_{\delta(\tau)}}\,\mathrm{d}\tau\Big)^{\frac{1}{2}}.$$

On fait alors l'hypothèse suivante sur la donnée initiale (dépendant de $\varepsilon \in [0,1]$) pour $\delta_{\text{in}} > 0$ et $\ell > \frac{d}{2}$ fixés :

Hypothèse 1. $\psi^\varepsilon_{\text{in}}, \phi^\varepsilon_{\text{in}} \in \mathcal{C}^1(\mathbb{R}^d)$ *sont tels que* $\nabla\psi^\varepsilon_{\text{in}} \in \mathcal{H}^\ell_{\delta_{\text{in}}}$ *et* $\nabla\phi^\varepsilon_{\text{in}} \in \mathcal{H}^{\ell+1}_{\delta_{\text{in}}}$ *sont bornés uniformément en* $\varepsilon \in [0,1]$ *dans ces espaces : il existe* ω_{in} *tel que pour tout* $\varepsilon \in [0,1]$

$$\|\nabla\psi^\varepsilon_{\text{in}}\|^2_{\mathcal{H}^\ell_{\delta_{\text{in}}}} + \|\nabla\phi^\varepsilon_{\text{in}}\|^2_{\mathcal{H}^{\ell+1}_{\delta_{\text{in}}}} \leq \omega_{\text{in}}.$$

On note par ailleurs :

$$(\tilde{D}_k^\varepsilon)^2 := \left\|\psi_{\text{in}}^\varepsilon - \psi_{\text{in}}^0\right\|_{\mathcal{H}_{\delta_{\text{in}}}^k}^2 + \left\|\phi_{\text{in}}^\varepsilon - \phi_{\text{in}}^0\right\|_{\mathcal{H}_{\delta_{\text{in}}}^{k+1}}^2.$$

On montre alors le résultat principal suivant :

Théorème 5.7. *Pour tout $(\psi_{\text{in}}^\varepsilon, \phi_{\text{in}}^\varepsilon)$ satisfaisant l'hypothèse 1, il existe $T > 0$, $M > 0$ et $\delta = \delta(t) := \delta_{\text{in}} - Mt$ tel que, pour tout $\varepsilon \in [0,1]$:*

- *Il existe une unique solution $(\psi^\varepsilon, \phi^\varepsilon) \in \mathcal{C}^2([0,T] \times \mathbb{R}^d)^2$ de (5.7) avec données initiales $(\psi^\varepsilon(0), \phi^\varepsilon(0)) = (\psi_{\text{in}}^\varepsilon, \phi_{\text{in}}^\varepsilon)$ tel que $(\nabla\psi^\varepsilon, \nabla\phi^\varepsilon) \in L^\infty((0,T), \mathcal{H}_\delta^\ell \times \mathcal{H}_\delta^{\ell+1}) \cap \mathcal{C}([0,T], \mathcal{H}_\delta^{\ell-\frac{1}{2}} \times \mathcal{H}_\delta^{\ell+\frac{1}{2}}) \cap L^2((0,T), \mathcal{H}_\delta^{\ell+\frac{1}{2}} \times \mathcal{H}_\delta^{\ell+\frac{3}{2}})$.*
- *Il existe une constante $C > 0$ indépendante de $\varepsilon \in [0,1]$ tel que*

$$\left|\!\left|\!\left|\psi^\varepsilon - \psi^0\right|\!\right|\!\right|_{\infty,T,\ell+\frac{1}{2},\delta} + \left|\!\left|\!\left|\phi^\varepsilon - \phi^0\right|\!\right|\!\right|_{\infty,T,\ell+\frac{3}{2},\delta} + \left|\!\left|\!\left|\psi^\varepsilon - \psi^0\right|\!\right|\!\right|_{2,T,\ell+1,\delta} + \left|\!\left|\!\left|\phi^\varepsilon - \phi^0\right|\!\right|\!\right|_{2,T,\ell+2,\delta} \leq C\Big(\sqrt{\varepsilon} + \tilde{D}_{\ell+\frac{1}{2}}^\varepsilon\Big),$$

et, si $\ell > \frac{d+1}{2}$,

$$\left|\!\left|\!\left|\psi^\varepsilon - \psi^0\right|\!\right|\!\right|_{\infty,T,\ell,\delta} + \left|\!\left|\!\left|\phi^\varepsilon - \phi^0\right|\!\right|\!\right|_{\infty,T,\ell+1,\delta} + \left|\!\left|\!\left|\psi^\varepsilon - \psi^0\right|\!\right|\!\right|_{2,T,\ell+\frac{1}{2},\delta} + \left|\!\left|\!\left|\phi^\varepsilon - \phi^0\right|\!\right|\!\right|_{2,T,\ell+\frac{3}{2},\delta} \leq C\Big(\varepsilon + \tilde{D}_\ell^\varepsilon\Big).$$

Notons que l'hypothèse d'analycité n'est que sur les dérivées spatiales $(\nabla\psi^\varepsilon, \nabla\phi^\varepsilon)$. Ceci est effectivement plus général que de supposer ψ^ε et ϕ^ε analytiques. En effet, ces derniers peuvent très bien avoir des limites différentes à l'infini, tendre vers l'infini, ou encore avoir une limite finie et une autre infinie en dimension $d = 1$ (voir Section 4.6).

Ces conclusions permettent également d'obtenir la convergence des observables quadratiques $|u^\varepsilon|^2$ et $\operatorname{Im}(\varepsilon\overline{u^\varepsilon}\nabla u^\varepsilon)$ dès lors que les données initiales convergent bien, dans l'un des deux sens suivants :

Hypothèse 2. *Il existe $C > 0$ tel que :*

$$\tilde{D}_{\ell+\frac{1}{2}}^\varepsilon \leq C\sqrt{\varepsilon}.$$

Hypothèse 3. *$\ell > \frac{d+1}{2}$ et il existe $C > 0$ tel que :*

$$\tilde{D}_\ell^\varepsilon \leq C\varepsilon.$$

Théorème 5.8. *Sous l'une des hypothèses 2 ou 3, les densités de position et de moment angulaire convergent dans le sens suivant : pour tout sous-ensemble compact $K \subset \mathbb{R}^d$, tout $k \in \mathbb{N}$ et tout $T' < \frac{\delta_{\text{in}}}{M}$ tel que $T' \leq T$, on a*

$$|u^\varepsilon|^2 \underset{\varepsilon\to 0}{\longrightarrow} e^{\psi^0}, \quad \text{et} \quad \operatorname{Im}(\varepsilon\overline{u^\varepsilon}\nabla u^\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} e^{\psi^0}\, v^0, \quad \text{dans } L^\infty((0,T'), \mathcal{C}^k(K)).$$

De plus, si tout les $\psi_{\text{in}}^\varepsilon(x)$ sont uniformément majorés, alors tous les $\psi^\varepsilon(t,x)$ sont uniformément majorés dans $(0,T') \times \mathbb{R}^d$ et on a pour tout $k \in \mathbb{N}$

$$\left|\!\left|\!\left||u^\varepsilon|^2 - e^{\psi^0}\right|\!\right|\!\right|_{L^\infty_{T'}H^k} + \left|\!\left|\!\left|\operatorname{Im}(\varepsilon\overline{u^\varepsilon}\nabla u^\varepsilon) - e^{\psi^0}\, v^0\right|\!\right|\!\right|_{L^\infty_{T'}H^k} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Cependant, ces considérations ne sont pas suffisantes pour obtenir l'asymptotique complète de la fonction d'onde u^ε. En effet, la convergence de ϕ^ε prouvée est au plus en ε, ce qui ne donne une asymptotique de $u^\varepsilon = e^{\frac{\psi^\varepsilon}{2} + i\frac{\phi^\varepsilon}{\varepsilon}}$ que en $O(1)$. Ceci a déjà été observé dans un cadre BKW plus classique d'une non-linéarité de type puissance ([27, 34]). Pour y remédier, il est alors nécessaire d'ajouter un terme correctif donnant une approximation de ϕ^ε en $o(\varepsilon)$. En effectuant un développement limité (formel) de (5.9) pour $\phi^\varepsilon = \phi^0 + \varepsilon\phi_1 + o(\varepsilon)$ et $\psi^\varepsilon = \psi^0 + \varepsilon\psi_1 + o(\varepsilon)$ et ne gardant que les termes d'ordre ε, on obtient alors le système suivant sur $v_1 = \nabla\phi_1$ et $\zeta_1 = \nabla\psi_1$:

$$\begin{cases} \partial_t v_1 + \nabla(v^0 \cdot v_1) + \lambda \operatorname{Re}\zeta_1 = 0, & v_1(0) = \nabla\phi_{\text{in},1}, \\ \partial_t\zeta_1 + \nabla(v^0 \cdot \zeta_1) + \nabla(v_1 \cdot \zeta^0) + \nabla \operatorname{div} v_1 = \dfrac{i}{2}\Big(\nabla \operatorname{div}\zeta^0 + 2\,\nabla(\zeta^0 \cdot \zeta^0)\Big), & \zeta_1(0) = \nabla\psi_{\text{in},1}, \end{cases}$$

et des relations similaires à (5.8). La théorie de Cauchy (locale) de ce système est relativement similaire au système précédent pour une donnée initiale $(\nabla\psi_{\text{in},1}, \nabla\phi_{\text{in},1}) \in \mathcal{H}_{\delta_{\text{in}}}^m \times \mathcal{H}_{\delta_{\text{in}}}^{m+1}$ avec $\frac{d-1}{2} < m \leq \ell - 1$. On prouve alors le résultat suivant montrant l'asymptotique complète de la fonction d'onde dès lors que les données initiales convergent correctement :

Théorème 5.9. *Supposons que $\ell > \frac{d+3}{2}$ et qu'il existe $(\psi_{\text{in},1}, \phi_{\text{in},1})$ tel que*

$$\tilde{r}_{\ell-1}^\varepsilon := \left\|\psi_{\text{in}}^\varepsilon - (\psi_{\text{in}}^0 + \varepsilon\psi_{\text{in},1})\right\|_{\mathcal{H}_{\delta_{\text{in}}}^{\ell-1}} + \left\|\phi_{\text{in}}^\varepsilon - (\phi_{\text{in}}^0 + \varepsilon\phi_{\text{in},1})\right\|_{\mathcal{H}_{\delta_{\text{in}}}^\ell} = o(\varepsilon) \qquad \text{as } \varepsilon \to 0.$$

Alors il existe une constante $C > 0$ tel que pour tout $\varepsilon \in [0,1]$,

$$\left|\left|\left|\psi^\varepsilon - (\psi^0 + \varepsilon\psi_1)\right|\right|\right|_{\infty,T,\ell-1,\delta} + \left|\left|\left|\psi^\varepsilon - (\psi^0 + \varepsilon\psi_1)\right|\right|\right|_{2,T,\ell-\frac{1}{2},\delta} \leq C(\tilde{r}^\varepsilon_{\ell-1} + \varepsilon^2),$$
$$\left|\left|\left|\phi^\varepsilon - (\phi^0 + \varepsilon\phi_1)\right|\right|\right|_{\infty,T,\ell,\delta} + \left|\left|\left|\phi^\varepsilon - (\phi^0 + \varepsilon\phi_1)\right|\right|\right|_{2,T,\ell+\frac{1}{2},\delta} \leq C(\tilde{r}^\varepsilon_{\ell-1} + \varepsilon^2).$$

En particulier, pour tout sous-ensemble $K \subset \mathbb{R}^d$ compact, tout $k \in \mathbb{N}$ et tout $T' < \frac{\delta_{\text{in}}}{M}$ tel que $T' \leq T$, on a :

$$\left\| u^\varepsilon - e^{\frac{\psi^0}{2} + i\phi_1 + i\frac{\phi^0}{\varepsilon}} \right\|_{L^\infty_{T'}\mathcal{C}^k(K)} = O\Big(\frac{r^\varepsilon_1}{\varepsilon} + \varepsilon\Big) \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

et, si $\psi^\varepsilon_{\text{in}}$ est uniformément majoré,

$$\left\| u^\varepsilon - e^{\frac{\psi^0}{2} + i\phi_1 + i\frac{\phi^0}{\varepsilon}} \right\|_{L^\infty_{T'}\mathcal{C}^k_b(\mathbb{R}^d)} = O\Big(\frac{r^\varepsilon_1}{\varepsilon} + \varepsilon\Big) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Une discussion sur l'apparition de ces termes correcteurs (ψ_1, ϕ_1) dans un cadre plus classique et sur le fait qu'ils soient triviaux (ou non) peut être trouvée dans [27]. Dans notre cadre particulier, on prouve assez facilement, en supposant toutes les données initiales $(\psi^\varepsilon_{\text{in}}, \phi^\varepsilon_{\text{in}})$ réelles, que ϕ_1 n'est nulle que si $(\psi_{\text{in},1}, \phi_{\text{in},1}) \equiv (0,0)$.

6. LISTE DES TRAVAUX RASSEMBLÉS DANS LA THÈSE

Les chapitres de ce manuscrit sont composés des travaux suivants :
- Chapitre 1 : article [67], à paraître dans *Analysis & PDE* (14:2).
- Chapitre 2 : article [69], paru dans *DCDS-A*.
- Chapire 3 : article [68], paru dans les *Annales de l'Institut Henri Poincaré, Analyse non-linéaire*.
- Chapitre 4 : article [70], pré-publication, soumis.

7. PERSPECTIVES

Quelques problèmes qu'il semble naturel d'envisager dans le prolongement de cette thèse sont brièvement présentés dans ce qui suit.

7.1. Convergence du rescaling dans le cas défocalisant

Même si les seules convergences de $|v^\varepsilon(t)|^2$ à $\varepsilon > 0$ fixé actuellement prouvées sont des convergences faibles, au mieux en distance de Wasserstein quadratique $\mathcal{W}_2$, on attendrait cependant une convergence forte dans L^1 tout comme pour les données gaussiennes. Une des pistes à exploiter serait d'utiliser l'énergie modifiée pour $|v^\varepsilon(t)|^2$, à savoir

$$\mathcal{E}^\varepsilon_{\text{kin}}(t) := \frac{\varepsilon^2}{2\,\tau(t)^2}\|\nabla v_\varepsilon\|^2_{L^2}, \qquad \mathcal{E}^\varepsilon_{\text{ent}}(t) := \int_{\mathbb{R}^d} |v_\varepsilon(t,y)|^2 \ln\left|\frac{v_\varepsilon(t,y)}{\gamma(y)}\right|^2 \mathrm{d}y,$$
$$\mathcal{E}^\varepsilon := \mathcal{E}^\varepsilon_{\text{kin}} + \lambda\,\mathcal{E}^\varepsilon_{\text{ent}}.$$

En effet, cette énergie satisfait

$$\dot{\mathcal{E}^\varepsilon} = -2\frac{\dot\tau(t)}{\tau(t)}\mathcal{E}^\varepsilon_{\text{kin}},$$

et est donc strictement décroissante. Comme il a déjà été remarqué dans [33], on aimerait idéalement pouvoir dire que cette énergie tend vers 0 lorsque $t \to \infty$. En effet, l'inégalité de Csiszár-Kullback ([7, Theorem 8.2.7]) peut ici être appliquée à $|v(t)|^2$ et γ^2 puisque $\| |v(t)| \|_{L^1} = \|\gamma^2\|_{L^1}$, donnant

$$\mathcal{E}^\varepsilon_{\text{ent}}(t) \geq \frac{1}{2\|\gamma^2\|_{L^1(\mathbb{R}^d)}}\left\| |v_\varepsilon(t)|^2 - \gamma^2 \right\|^2_{L^1(\mathbb{R}^d)},$$

ce qui permettrait de conclure. Malheureusement, la seule conclusion que l'on peut *a priori* inférer de l'équation d'évolution précédente est qu'il existe une suite $t_k \to \infty$ tel que $\mathcal{E}^\varepsilon_{\text{kin}}(t_k) \underset{k\to\infty}{\longrightarrow} 0$.

Une autre problématique intéressante serait d'obtenir un taux de convergence explicite pour la distance de Wasserstein quadratique $\mathcal{W}_2$, si possible indépendante de ε. On s'attendrait, au vu des résultats de convergence sur les moments et du cas particulier de données gaussiennes, à avoir un taux de convergence également en $\sqrt{\ln t}^{-1}$ tout comme pour $\mathcal{W}_1$.

7.2. Convergence pour les multi-gaussiennes

La convergence obtenue pour les multi-gaussiennes n'est actuellement qu'en norme L^2, les techniques d'énergie ne pouvant être appliquées à des gaussiennes qui ne sont pas des points critiques de l'énergie. Pour autant, au vu des simulations numériques, il serait assez naturel d'avoir également une convergence dans l'espace d'énergie $W(\mathbb{R}^d)$, probablement avec la même vitesse de convergence que celle dans L^2. Par ailleurs, si on avait cette convergence (ou juste dans H^1 au moins), on devrait pouvoir prouver (de la même manière que pour les multi-Gaussons) la convergence dans $\mathcal{F}(H^1)$, ce qui donnerait ainsi une convergence dans $H^1 \cap \mathcal{F}(H^1)$ exactement comme pour les multi-Gaussons.

7.3. Stabilité des multi-Gaussons et multi-gaussiennes

Dans la continuité de l'existence de multi-Gaussons, l'étape suivante est naturellement la stabilité (asymptotique) de ces solutions. Tout comme pour l'existence, la singularité du logarithme est problématique pour pouvoir appliquer les résultats de stabilité de multi-solitons sur NLS. L'adaptation de ces derniers semble cependant beaucoup plus compliquée que ceux d'existence, d'autant plus que le résultat de rigidité (5.5) donne une condition qui est *a priori* peu habituelle : on penserait en effet que le taux de convergence exponentielle devrait être fonction de la vitesse relative v_* entre les solitons, et devrait ne pas être majoré lorsque $|v_*| \to \infty$, ce qui est contredit par ce résultat de rigidité.

Parallèlement, la stabilité des multi-gaussiennes semble également un problème intéressant. On attendrait que celles-ci ne soient pas (asymptotiquement) stables puisque n'étant même pas des points critiques de l'énergie. Malgré tout, la singularité du logarithme pourrait bien donner des comportements beaucoup plus inhabituels. Les simulations numériques semblent par ailleurs aller dans ce sens.

7.4. Généralisation à d'autres non-linéarités et équations

Une question intéressante serait de se demander le comportement de la solution à NLS (2.1) où la non-linéarité est une somme d'un logarithme et d'une puissance (sous-critique H^1 pour commencer) :

$$f(x) = \lambda \ln x + \mu x^{\frac{\alpha-1}{2}}.$$

Lorsque $\lambda < 0$ et $\mu < 0$, il semble normal qu'on ait un régime défocalisant : le comportement de la solution en temps grand serait donc plutôt fonction du comportement de la non-linéarité proche de 0, qui croît logarithmiquement vers $+\infty$. On se doute donc que le comportement est similaire à (3.1) dans le cas défocalisant. Ces remarques sont d'ailleurs rigoureusement prouvées dans l'article de R. Carles et I. Gallagher ([33]).

Inversement, si $\lambda > 0$ et $\mu > 0$, on devrait se retrouver dans un régime focalisant. Le comportement à l'infini des solitons de cette équation, tendant vers 0 à l'infini, ne dépendraient également que du comportement de la non-linéarité proche de 0, et donc avoir une décroissance plus forte que exponentielle, probablement une décroissance "gaussienne", de la même manière que pour (3.1). Si on reprend le point de vue physique d'une énergie potentielle à minimiser, on pourrait se douter que le comportement soit par ailleurs similaire dans le cas $\lambda > 0$ et $\mu < 0$, à condition d'avoir une bonne théorie de Cauchy, ce qui implique donc d'avoir une puissance sous-critique H^1 (ou au plus critique).

Dans ces trois cas, des résultats similaires à ceux de cette thèse devraient être accessibles en adaptant les preuves. Cependant, lorsque $\lambda < 0$ et $\mu > 0$ ont des signes différents, le comportement risque d'être bien plus compliqué. Même lorsqu'on reprend le point de vue de l'énergie potentielle à minimiser, celle-ci peut être minimisée à la fois en 0 et à l'infini, il serait donc nécessaire d'effectuer une étude plus poussée pour mieux comprendre le comportement dans ce cas.

De manière plus générale, on pourrait se poser la question du comportement d'une non-linéarité singulière en 0 mais tendant vers l'infini moins rapidement (ou au plus aussi rapidement) que le logarithme. Si la dérivée de f est bornée par $\frac{1}{x}$ (au moins localement autour de 0, à une constante multiplicative près), l'estimation d'énergie L^2 du Lemme 3.1 devrait pouvoir être toujours valable dans ce cas. Lorsqu'on se retrouve dans un cas focalisant (c'est-à-dire lorsque $f(x) \to -\infty$), on devrait se retrouver également avec une décroissance des solitons plus fortes qu'exponentielle à l'infini, et une tactique similaire à celle de cette thèse devrait pouvoir être dévelopée pour obtenir une existence de multi-solitons. Dans un régime défocalisant en revanche, on s'attend à avoir une dispersion dont la vitesse devrait être entre la vitesse de dispersion linéaire $t^{\frac{d}{2}}$ et celle pour logNLS $(t\sqrt{\ln t})^{\frac{d}{2}}$.

Une autre approche consisterait à rajouter un terme linéaire avec un potentiel quadratique. Pour une telle équation, les données gaussiennes resteront également gaussiennes. Cependant, les EDO caractérisant les paramètres de ces gaussiennes vont changer, et une première étape consisterait à analyser ces EDO pour comprendre le comportement des solutions gaussiennes, que l'on peut par la suite étendre aux solutions générales.

Indépendamment, la non-linéarité logarithmique se retrouve dans de nombreuses autres équations. Une généralisation des résultats de cette thèse à ces équations (on pense notamment à logKdV) est une autre perspective à étudier.

Chapitre 1

Convergence rate in Wasserstein distance and semiclassical limit for the defocusing logarithmic Schrödinger equation

Abstract. We consider the dispersive logarithmic Schrödinger equation in a semiclassical scaling. We extend the results of [33] about the large-time behavior of the solution (dispersion faster than usual with an additional logarithmic factor and convergence of the rescaled modulus of the solution to a universal Gaussian profile) to the case with semiclassical constant. We also provide a sharp convergence rate to the Gaussian profile in the Kantorovich-Rubinstein metric through a detailed analysis of the Fokker-Planck equation satisfied by this modulus. Moreover, we perform the semiclassical limit of this equation thanks to the Wigner Transform in order to get a (Wigner) measure. We show that those two features are compatible and the density of a Wigner Measure has the same large time behavior as the modulus of the solution of the logarithmic Schrödinger equation. Lastly, we discuss about the related kinetic equation (which is the *Kinetic Isothermal Euler System*) and its formal properties, enlightened by the previous results and a new class of explicit solutions.

1.1. Introduction

1.1.1. Setting

We are interested in the *Logarithmic Non-Linear Schrödinger Equation* with semiclassical constant

$$i\varepsilon\,\partial_t u_\varepsilon + \frac{\varepsilon^2}{2}\Delta u_\varepsilon = \lambda u_\varepsilon \ln |u_\varepsilon|^2, \qquad u_{\varepsilon|_{t=0}} = u_{\varepsilon,\text{in}}, \tag{1.1.1}$$

with $x \in \mathbb{R}^d$, $d \geq 1$, $\lambda \in \mathbb{R} \setminus \{0\}$, $\varepsilon > 0$. It was introduced as a model of nonlinear wave mechanics and in nonlinear optics ([23], see also [25, 95, 98, 102, 59]). The case $\lambda < 0$ is interesting from a physical point of view and has been studied formally and rigorously without semiclassical constant (i.e. $\varepsilon = 1$, see [58, 98]). On the other hand, R. Carles and I. Gallagher recently went further in the case $\lambda > 0$ (also with $\varepsilon = 1$) whose study goes back to [40, 86]. After improving the result of [86] for the Cauchy problem, they proved not only that this case is actually the defocusing case with an unusually faster dispersion but also that a universal behavior occurs: up to a rescaling, the modulus of the solution converges to a universal Gaussian profile (see [33]).

In the context of (non-linear) Schrödinger equations, a usual question is the behavior of the solution when $\varepsilon \to 0$ known as the *semiclassical limit*, making the link between quantum mechanics and classical mechanics in physics. It has also been studied a lot in mathematics in order to get a good and rigorous framework for reaching the limit. Indeed, u_ε typically does not have a meaningful limit and that is the reason why several asymptotic techniques have been developed to treat semiclassical (also called *high-frequency*, or *short-wavelength* in some contexts) problems. One of the most powerful and elegant tools was introduced by Wigner ([154]) in 1932. Known nowadays as the Wigner Transform, it has been analyzed a lot ([110, 10, 88, 87] for instance) and usually allows a simple and nice description of the semiclassical limit. For any sequence of functions $f_\varepsilon = f_\varepsilon(x) \in L^2(\mathbb{R}^d)$ for $\varepsilon > 0$, the Wigner Transform W_ε defined by

$$W_\varepsilon(x,\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\xi\cdot z} f_\varepsilon\left(x + \frac{\varepsilon z}{2}\right)\overline{f_\varepsilon\left(x - \frac{\varepsilon z}{2}\right)}\,\mathrm{d}z, \qquad (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d,$$

is a real-valued function on the phase space. It is known that under suitable assumptions, up to a subsequence, this function converges weakly to a measure, called Wigner measure; see e.g. [110, 88]. Moreover, if u_ε satisfies $i\varepsilon\,\partial_t u_\varepsilon + \frac{\varepsilon^2}{2}\Delta u_\varepsilon = V_0 u_\varepsilon$ and if the potential V_0 is smooth enough then the Wigner Measure $W(t)$ of $(u_\varepsilon(t))_{\varepsilon>0}$ satisfies the Vlasov (or kinetic) equation

$$\partial_t W + \xi \cdot \nabla_x W + \nabla_x V_0 \cdot \nabla_\xi W = 0.$$

As a follow-up of [33], this article has two main purposes: reaching the semiclassical limit thanks to the Wigner Transform and computing the convergence rate to the Gaussian profile in Wasserstein distance. Actually, those two features are compatible since the convergence rate is actually independent of $\varepsilon \in (0,1]$ and then goes through the limit $\varepsilon \to 0$ under suitable assumptions. This is a very interesting and rare feature: it has been shown that the large time behavior and the semiclassical limit do not usually commute, for instance for linear Schrödinger equations with potential (see [51, 90, 92, 157, 159]). Moreover, in general, Wigner measures are not a suitable tool to address nonlinear problems, except in the case of the Schrödinger-Poisson equation (see [162, 32]). On the other hand, at least in the case $\varepsilon = 1$, (1.1.1) exhibits rather strong nonlinear effects (modified dispersion, universal asymptotic profile), so it is rather surprising that such a result can be established.

1.1.2. Universal dynamics without semiclassical constant

Throughout the rest of this paper, we assume $\lambda > 0$. We recall the *Logarithmic Non-Linear Schrödinger Equation* without semiclassical constant ($\varepsilon = 1$)

$$i\,\partial_t u + \frac{\Delta u}{2} = \lambda u \ln|u|^2, \qquad u(0,.) = u_{\text{in}}. \tag{1.1.2}$$

Following the notations used in [33], for $0 < \alpha \leq 1$, we define

$$\mathcal{F}(H^\alpha) := \left\{ u \in L^2(\mathbb{R}^d), x \mapsto \langle x\rangle^\alpha u(x) \in L^2(\mathbb{R}^d) \right\},$$

where $\langle x\rangle = \sqrt{1+|x|^2}$ and $\mathcal{F}$ is the Fourier Transform. $\mathcal{F}(H^\alpha)$ is endowed with its natural norm. In the same way, we also define the mass, the angular momentum and the energy (with semiclassical constant) for all $f \in \left\{ g \in H^1(\mathbb{R}^d), |g|^2 \ln|g|^2 \in L^1 \right\}$:

$$M(f) := \|f\|_{L^2}, \qquad \mathcal{J}_\varepsilon(f) := \varepsilon \operatorname{Im} \int_{\mathbb{R}^d} \overline{f(x)}\, \nabla f(x)\, \mathrm{d}x,$$

$$E_\varepsilon(f) := \frac{\varepsilon^2}{2} \|\nabla f\|_{L^2} + \lambda \int_{\mathbb{R}^d} |f(x)|^2 \ln|f(x)|^2\, \mathrm{d}x.$$

The Cauchy problem is investigated in [86] and improved by [33, Theorem 1.5.], showing well-posedness for initial data in $H^1 \cap \mathcal{F}(H^\alpha)(\mathbb{R}^d)$ with $0 < \alpha \leq 1$ and conservation of those three quantities (with $\varepsilon = 1$). Then, in the same paper, the authors studied large time behavior of the solution when $\alpha = 1$. Two features characterizing the dynamics associated to (1.1.2) are unusual:

- The dispersion is in $\left(t\sqrt{\ln t}\right)^{\frac{d}{2}}$. Usually in $t^{\frac{d}{2}}$ for the Schrödinger equation, it is altered by a logarithmic factor due to the non-linearity of the equation, accelerating the dispersion.
- Up to a rescaling, the modulus of the solution converges for large time to a universal Gaussian profile weakly in L^1.

Those aspects are stated in the main theorem of [33], recalled in Theorem 1.1.1. Denote by $\gamma(x) := e^{-\frac{|x|^2}{2}}$ for $x \in \mathbb{R}^d$ and $\tau \in \mathcal{C}^\infty(\mathbb{R})$ the solution to

$$\ddot{\tau} = \frac{2\lambda}{\tau}, \qquad \tau(0) = 1, \qquad \dot{\tau}(0) = 0. \tag{1.1.3}$$

This function satisfies $\tau(t) \sim 2t\sqrt{\lambda \ln t}$ and $\dot{\tau}(t) \sim 2\sqrt{\lambda \ln t}$ as $t \to +\infty$ (see [33, Lemma 1.6.]). We also define the following quantities (with semiclassical constant) for any function $f \in \mathcal{F}(H^1) \cap H^1(\mathbb{R}^d)$:

$$\tilde{E}_\varepsilon\big(t,f\big) := \int_{\mathbb{R}^d} \left(|y|^2 + \left|\ln|f|^2\right|\right) |f|^2\, \mathrm{d}y + \frac{\varepsilon^2}{\tau(t)^2} \|\nabla_y f\|^2_{L^2(\mathbb{R}^d)}, \qquad \tilde{E}^0_\varepsilon(f) = \tilde{E}_\varepsilon\big(0,f\big). \tag{1.1.4}$$

Theorem 1.1.1 ([33, Theorem 1.7.]). *Let $\lambda > 0$ and $u_{\text{in}} \in \mathcal{F}(H^1) \cap H^1(\mathbb{R}^d) \setminus \{0\}$. Rescale the solution provided by [33, Theorem 1.5.] to $v = v(t,y)$ by setting*

$$u(t,x) = \frac{1}{\tau(t)^{\frac{d}{2}}} \frac{\|u_{\text{in}}\|_{L^2}}{\|\gamma\|_{L^2}}\, v\left(t, \frac{x}{\tau(t)}\right) e^{i\, \frac{\dot{\tau}(t)}{\tau(t)} \frac{|x|^2}{2}}.$$

Then there exists C such that for all $t \geq 0$,

$$\tilde{E}_1\big(t, v(t)\big) \leq C.$$

We have moreover

$$\int_{\mathbb{R}^d} |y|^2 |v(t,y)|^2\, \mathrm{d}y \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} |y|^2 \gamma^2(y)\, \mathrm{d}y.$$

Finally,

$$|v(t,.)|^2 \underset{t\to\infty}{\rightharpoonup} \gamma^2 \qquad \textit{weakly in } L^1(\mathbb{R}^d).$$

Remark 1.1.2. As a straightforward consequence, with the notations of the previous theorem, $|v(t)|^2$ converges to γ^2 in Wasserstein distance:

$$\mathcal{W}_2\left(\pi^{-\frac{d}{2}}\,|v(t)|^2, \pi^{-\frac{d}{2}}\,\gamma^2\right) \underset{t\to\infty}{\longrightarrow} 0,$$

where we recall that the Wasserstein distance is defined for ν_1 and ν_2 probability measures by

$$\mathcal{W}_p(\nu_1,\nu_2) = \inf\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p \,\mathrm{d}\mu(x,y)\right)^{\frac{1}{p}}; \quad (\pi_j)_\#\mu = \nu_j\right\},$$

where μ varies among all probability measures on $\mathbb{R}^d\times\mathbb{R}^d$, and $\pi_j : \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d$ denotes the canonical projection onto the j-th factor (see e.g. [145]).

Remark 1.1.3. Another consequence, stated in [33, Corollary 1.10.], is the logarithmic growth of the Sobolev norms of the solution as soon as the initial data is not null:

$$\|\nabla u(t)\|_{L^2}^2 \underset{t\to+\infty}{\sim} 2\lambda d\|u_{\mathrm{in}}\|_{L^2}\,\ln t, \qquad \text{and} \qquad (\ln t)^{\frac{\delta}{2}} \lesssim \|u(t)\|_{\dot{H}^\delta} \lesssim (\ln t)^{\frac{\delta}{2}}, \qquad \forall t>1, \forall\delta\in(0,1),$$

where $\dot{H}^\delta(\mathbb{R}^d)$ denotes the standard homogeneous Sobolev space.

The weak convergence in L^1 found in [33, Theorem 1.7.] actually comes from the fact that, after a change of time variable, $\rho(t) = |v(t,.)|^2$ satisfies a Fokker-Planck equation with some source terms which are negligible (in some way) when $t\to\infty$, along with the compactness of $\{\rho(t), t\in\mathbb{R}\}$ in L^1_w. To provide this weak convergence, the authors first take the limit $t\to+\infty$ (up to a subsequence) and then use the properties of this Fokker-Planck equation along with the fact that the limit satisfies the same Fokker-Planck equation without source term to conclude that the limit is a universal Gaussian profile (and that the whole sequence converges).

However, the Fokker-Planck operator $L = \Delta + \nabla\cdot(2y\,.)$ is extremely particular. Indeed, unlike most of the other Fokker-Planck operators, its form allows to compute explicitly its kernel, which leads to better estimates for the solution. Those estimates are helpful in order to compute some convergence rate. For this, we have to consider a distance which metrizes the weak convergence in L^1 (no strong convergence has been proved). Since there is also convergence of the first two momenta, we focus on the Wasserstein metric, and mostly on the 1-Wasserstein distance (also called the Kantorovich-Rubinstein metric) because the Kantorovich-Rubinstein duality gives an easier framework.

1.1.3. Main results

Universal dynamics with semiclassical constant. Introducing the semiclassical constant in the equation, we now want to investigate (1.1.1). First of all, we need to face the Cauchy problem, which is easy to state thanks to [33, Theorem 1.5.].

Theorem 1.1.4. *Given any $\varepsilon>0$, $\lambda>0$ and any initial data $u_{\varepsilon,\mathrm{in}}\in H^1\cap\mathcal{F}(H^\alpha)(\mathbb{R}^d)$ with $0<\alpha\le 1$, there exists a unique, global solution $u_\varepsilon\in L^\infty_{loc}\left(\mathbb{R}, H^1\cap\mathcal{F}(H^\alpha)(\mathbb{R}^d)\right)\cap\mathcal{C}\left(\mathbb{R}, H^{-1}\cap L^2_w(\mathbb{R}^d)\right)$ to (1.1.1). Moreover, the mass $M(u_\varepsilon(t))$, the angular momentum $\mathcal{J}_\varepsilon(u_\varepsilon(t))$ and the energy $E_\varepsilon(u_\varepsilon(t))$ are independent of time.*

In the same way as in [33], the first main focus of this paper concerns large time behavior of this solution. The results of Theorem 1.1.1 can be extended as well, and the same features as without semiclassical constant hold (faster dispersion with a logarithmic factor and convergence to a universal Gaussian profile after rescaling). But the main new feature of this result is the convergence rate to the Gaussian profile.

Theorem 1.1.5. *Let $\lambda>0$, $\varepsilon>0$ and $u_{\varepsilon,\mathrm{in}}\in H^1\cap\mathcal{F}(H^1)(\mathbb{R}^d)\setminus\{0\}$. Rescale the solution u_ε provided by Theorem 1.1.4 to $v_\varepsilon = v_\varepsilon(t,y)$ by setting*

$$u_\varepsilon(t,x) = \frac{1}{\tau(t)^{\frac{d}{2}}}\,\frac{\|u_{\varepsilon,\mathrm{in}}\|_{L^2}}{\|\gamma\|_{L^2}}\,v_\varepsilon\left(t,\frac{x}{\tau(t)}\right)\,e^{i\,\frac{\dot{\tau}(t)}{\tau(t)}\frac{|x|^2}{2\,\varepsilon}}, \qquad v_{\varepsilon,in} := v_\varepsilon(0) = \frac{\|\gamma\|_{L^2}}{\|u_{\varepsilon,\mathrm{in}}\|_{L^2}}u_{\varepsilon,\mathrm{in}}. \tag{1.1.5}$$

There exists a non-decreasing continuous function $C:[0,\infty)\to[0,\infty)$ depending only on λ and d such that for all $t\ge 0$ and all $\varepsilon>0$,

$$\tilde{E}_\varepsilon\big(t,v(t)\big) \le C\left(\tilde{E}^0_\varepsilon(v_{\varepsilon,in})\right), \tag{1.1.6}$$

$$\int_0^\infty \frac{\varepsilon^2\,\dot{\tau}(t)}{\tau^3(t)}\|\nabla_y v_\varepsilon(t)\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}t \le C\left(\tilde{E}^0_\varepsilon(v_{\varepsilon,in})\right). \tag{1.1.7}$$

Moreover, the first two momenta converge: for all $t\ge 1$ and all $\varepsilon>0$,

$$\int_{\mathbb{R}^d} y\,|v_\varepsilon(t,y)|^2\,\mathrm{d}y = \frac{1}{\tau(t)}\,\frac{\|\gamma^2\|_{L^1}}{\|u_{\varepsilon,\mathrm{in}}\|^2_{L^2}}(I^\varepsilon_{1,0}\,t + I^\varepsilon_{2,0}) \underset{t\to\infty}{\longrightarrow} 0, \tag{1.1.8}$$

$$\left|\int_{\mathbb{R}^d} |y|^2\,|v_\varepsilon(t,y)|^2\,\mathrm{d}y - \int_{\mathbb{R}^d} |y|^2\,\gamma^2(y)\,\mathrm{d}y\right| \leq C\Big(\tilde{E}^0_\varepsilon(v_{\varepsilon,in})\Big)\,\frac{\dot{\tau}(t)+1}{\dot{\tau}(t)^2} \underset{t\to\infty}{\longrightarrow} 0, \tag{1.1.9}$$

where

$$I^\varepsilon_{1,0} = \operatorname{Im}\varepsilon\int_{\mathbb{R}^d} \overline{u_{\varepsilon,\text{in}}}\,\nabla u_{\varepsilon,\text{in}}\,\mathrm{d}y, \qquad I^\varepsilon_{2,0} = \int_{\mathbb{R}^d} y\,|u_{\varepsilon,\text{in}}|^2\,\mathrm{d}y.$$

Lastly, for all $t \geq 2$ *and all* $\varepsilon > 0$*,*

$$\mathcal{W}_1\left(\frac{|v_\varepsilon(t,.)|^2}{\pi^{\frac{d}{2}}}, \frac{\gamma^2}{\pi^{\frac{d}{2}}}\right) \leq C\Big(\tilde{E}^0_\varepsilon(v_{\varepsilon,in})\Big)\,\frac{1}{\sqrt{\ln t}}. \tag{1.1.10}$$

It is worth noticing that the bounds and convergence rates for v_ε depend on ε only through $\tilde{E}^0_\varepsilon(v_{\varepsilon,in})$. In particular, if we take suitable $u_{\varepsilon,\text{in}}$ such that $\tilde{E}^0_\varepsilon(v_{\varepsilon,in})$ is bounded, then all of them may be taken *independent of* $\varepsilon \in (0,1]$. This is a very important feature, which rarely happens for large time behavior in the context of the semiclassical limit.

If we also want uniform bounds and convergence rates for u_ε thanks to (1.1.5), $\|u_{\varepsilon,\text{in}}\|_{L^2}$ must be bounded. Thus, we introduce Assumption (A1):

$$(u_{\varepsilon,\text{in}})_{\varepsilon>0} \text{ uniformly bounded in } L^2(\mathbb{R}^d) \qquad \text{and} \qquad \Big(\tilde{E}^0_\varepsilon(v_{\varepsilon,in})\Big)_{\varepsilon\in(0,1]} \text{ is bounded,} \tag{A1}$$

where $v_{\varepsilon,in}$ is defined by (1.1.5). If those assumptions are satisfied, then all the bounds and convergence rates are *uniform* in $\varepsilon \in (0,1]$. Such a thing occurs for instance for WKB states:

$$\begin{aligned} &u_{\varepsilon,\text{in}} = \sqrt{\rho_{\text{in}}}\,e^{i\,\frac{\phi_{\text{in}}}{\varepsilon}}, \qquad \forall\varepsilon\in(0,1], \qquad \text{for some } \rho_{\text{in}} = \rho_{\text{in}}(x) \geq 0 \text{ and } \phi_{\text{in}} = \phi_{\text{in}}(x) \text{ such that:} \\ &\sqrt{\rho_{\text{in}}} \in \mathcal{F}(H^1)\cap H^1(\mathbb{R}^d)\setminus\{0\}, \qquad \phi_{\text{in}} \in W^{1,1}_{\text{loc}}(\mathbb{R}^d), \qquad \sqrt{\rho_{\text{in}}}\,\nabla\phi_{\text{in}} \in L^2(\mathbb{R}^d), \end{aligned} \tag{A2}$$

because those assumptions imply $\rho_{\text{in}}\ln\rho_{\text{in}} \in L^1(\mathbb{R}^d)$. Indeed, the two estimates

$$\int \rho_{\text{in}}^{1+\delta} \leq C_\delta\|\sqrt{\rho_{\text{in}}}\|^{1+\delta}_{H^1}$$

for $\delta > 0$ small enough thanks to Sobolev embeddings and

$$\int \rho_{\text{in}}^{1-\delta} \leq C_\delta\|\sqrt{\rho_{\text{in}}}\|^{2-2\delta-d\delta}_{L^2}\|\,|x|\sqrt{\rho_{\text{in}}}\,\|^{d\delta}_{L^2} \tag{1.1.11}$$

for $0 < \delta < \frac{2}{d+2}$ which can be readily proved by an interpolation method (cutting the integral into $|y| < R$ and $|y| > R$, using Hölder inequality and optimizing over R; see e.g. [35]) yield $\int \rho_{\text{in}}\,|\ln\rho_{\text{in}}\,|\,dy < \infty$. Moreover in such a case, $\|u_{\varepsilon,\text{in}}\|_{L^2}$, $I^\varepsilon_{1,0}$ and $I^\varepsilon_{2,0}$ are independent of ε.

The assumptions (A2) are well known as *WKB* states and the corresponding Wigner Measure (without time-dependence) is a *monokinetic* measure (see [110, Exemple III.5.]). Under stronger assumptions on ρ_{in} and ϕ_{in}, this feature usually propagates in time for some (non-linear) Schrödinger equations and we recover time-dependent monokinetic measure (see for instance [27]). However, it might be difficult to prove it for (1.1.1), except in a particular case (see Section 1.5.2).

Remark 1.1.6. The rescaling (1.1.5) is similar to that in Theorem 1.1.1 when adding the semiclassical constant: the main complex oscillations are altered by an ε^{-1} factor.

Remark 1.1.7. The convergence in Wasserstein distance is not new, we already know that we had convergence even with respect to $\mathcal{W}_2$ (at least for $\varepsilon = 1$). Yet, the convergence rate is an interesting new feature: no convergence rate (except for the momenta) was proven in [33]. Moreover, such a convergence rate is optimal in this way: if $I^\varepsilon_{1,0} \neq 0$ (which is often verified, unless the initial data are well prepared), the convergence rate of the first moment reads:

$$\int_{\mathbb{R}^d} y\,|v_\varepsilon(t,y)|^2\,\mathrm{d}y \underset{t\to\infty}{\sim} \frac{\|\gamma^2\|_{L^1}}{\|u_{\varepsilon,\text{in}}\|^2_{L^2}}\,\frac{I^\varepsilon_{1,0}}{2\sqrt{\lambda\ln t}}.$$

Therefore we cannot have a better convergence rate, at least in the general case.

Thanks to the bounds on the L^1 norm of $|v_\varepsilon(t,.)|^2$ and on its second momentum, the following corollary also holds:

Corollary 1.1.8. *With the notations of Theorem 1.1.5, for all* $t \geq 2$*, all* $\varepsilon > 0$ *and all* $\delta \in (0,1)$*,*

$$\begin{aligned} \big\||v_\varepsilon(t,.)|^2 - \gamma^2\big\|_{W^{-1+\delta,1}} &\leq C\Big(\tilde{E}^0_\varepsilon(v_{\varepsilon,in})\Big)\,\frac{1}{(\ln t)^{\frac{1-\delta}{2}}}, \\ \mathcal{W}_{1+\delta}\left(\frac{|v_\varepsilon(t,.)|^2}{\pi^{\frac{d}{2}}}, \frac{\gamma^2}{\pi^{\frac{d}{2}}}\right) &\leq C\Big(\tilde{E}^0_\varepsilon(v_{\varepsilon,in})\Big)\,\frac{1}{(\ln t)^{\frac{1-\delta}{2}}}. \end{aligned}$$

Finally, the Sobolev norms of all solutions grow in the same way as in [33], with in addition the semiclassical constant.

Corollary 1.1.9. *Given any $\varepsilon > 0$ and $\lambda > 0$, let $u_{\varepsilon,\mathrm{in}} \in H^1 \cap \mathcal{F}(H^1)(\mathbb{R}^d) \setminus \{0\}$. The solution u_ε to* (1.1.1) *satisfies as $t \to \infty$,*

$$\varepsilon^2 \|\nabla u_\varepsilon\|_{L^2}^2 \sim 2\lambda d \|u_{\varepsilon,\mathrm{in}}\|_{L^2}^2 \ln t,$$

and for all $\delta \in (0,1)$,

$$(\ln t)^{\frac{\delta}{2}} \lesssim \varepsilon^\delta \|u_\varepsilon\|_{\dot{H}^\delta} \lesssim (\ln t)^{\frac{\delta}{2}},$$

where $\dot{H}^\delta$ denotes the standard homogeneous Sobolev space.

The semiclassical limit. Following the previous remarks, we now want to study the semiclassical limit of (1.1.1), and the Wigner Transform is a natural tool we may use along with the usual space of test functions:

$$\mathcal{A} = \left\{\phi \in \mathcal{C}_0(\mathbb{R}_x^d \times \mathbb{R}_\xi^d), (\mathcal{F}_\xi \phi)(x,z) \in L^1(\mathbb{R}_z^d, \mathcal{C}_0(\mathbb{R}_x^d))\right\}$$

endowed with its natural norm which makes it a Banach space and algebra. In a lot of cases, the Wigner Transform converges pointwise in time to the Wigner Measure, which is continuous in time with values in $\mathcal{A}'$ and satisfies the linked kinetic (or Vlasov-type) equation, with the same potential (see for instance [110, 10, 88]). If a lot of potentials satisfy the assumptions of one of those results, this is not the case here (to the best of our knowledge). Indeed, our potential depends on the solution and is highly singular in the same time.

However, the framework given by Theorems 1.1.4 and 1.1.5 for our equation is still interesting for the Wigner Transform, considering that the solutions $(u_\varepsilon(t))_{\varepsilon>0}$ at time $t \in \mathbb{R}$ satisfy the usual assumptions in order to reach the limit for the Wigner Transform and get good properties for the Wigner Measure (see for example [110, Proposition III.1. and Théorème III.1.]). Therefore, an interesting framework would be to work in L^p locally in time for all $p < \infty$, even if it means losing the pointwise convergence of the Wigner Transform and the continuity of the Wigner Measure in time.

Before stating the theorem for the semiclassical limit, we denote by $\mathcal{M}(\mathbb{R}^d)$ the set of non-negative finite measures on $\mathbb{R}^d$, $\mathcal{P}(\mathbb{R}^d)$ the set of all probability measures and we also define

$$\begin{aligned} L_2^1(\mathbb{R}^d) &:= \left\{f \in L^1(\mathbb{R}^d), \int_{\mathbb{R}^d} |y|^2 \,|f(y)| \,\mathrm{d}y < \infty\right\}, \\ L\log L(\mathbb{R}^d) &:= \left\{f \in L^1(\mathbb{R}^d), |f| \log|f| \in L^1(\mathbb{R}^d)\right\}, \\ \mathcal{P}_i(\mathbb{R}^d) &:= \left\{\mu \in \mathcal{P}(\mathbb{R}^d), \int |x|^i \,\mathrm{d}\mu < \infty\right\} \qquad \text{endowed with } \mathcal{W}_i \text{ for } i = 1,2. \end{aligned}$$

Theorem 1.1.10. *Given any $\lambda > 0$ and $u_{\varepsilon,\mathrm{in}} \in H^1 \cap \mathcal{F}(H^1)(\mathbb{R}^d) \setminus \{0\}$ for all $\varepsilon > 0$ such that $(u_{\varepsilon,\mathrm{in}})_{\varepsilon>0}$ satisfies* (A1)*, define u_ε and v_ε provided by Theorem 1.1.5 for all $\varepsilon > 0$, and W_ε (resp. $\tilde{W}_\varepsilon$) the Wigner Transform of u_ε (resp. v_ε). Then there exists a subsequence $(\varepsilon_n)_n$ such that $\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$ and two (non-negative) finite measures W and $\tilde{W}$ in $L^\infty((0,\infty), \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d))$ such that for every $p \in [1,\infty)$*

$$W_{\varepsilon_n} \underset{n\to\infty}{\rightharpoonup} W \quad \text{in } L^p_{loc}((0,\infty), \mathcal{A}'), \qquad \tilde{W}_{\varepsilon_n} \underset{n\to\infty}{\rightharpoonup} \tilde{W} \quad \text{in } L^p_{loc}((0,\infty), \mathcal{A}'),$$

and the relation between W_ε and $\tilde{W}_\varepsilon$ given by

$$W_\varepsilon(t,x,\xi) = \frac{\|u_{\varepsilon,\mathrm{in}}\|_{L^2}^2}{\|\gamma^2\|_{L^1}} \tilde{W}_\varepsilon\left(t, \frac{x}{\tau(t)}, \tau(t)\,\xi - \dot{\tau}(t)\,x\right) \tag{1.1.12}$$

still holds after passing to the limit since $\|u_{\varepsilon_n,\mathrm{in}}\|_{L^2}$ converges (to some $M_0 \geq 0$) as $n \to \infty$. Furthermore, we have

$$\begin{aligned} \pi^{-\frac{d}{2}} \tilde{\rho}(t,y) := \pi^{-\frac{d}{2}} \int_{\mathbb{R}^d} \tilde{W}(t,y,\mathrm{d}\eta) \in L^\infty((0,\infty), L_2^1 \cap L\log L(\mathbb{R}^d)) \cap \mathcal{C}(\mathbb{R}^+, \mathcal{P}_1(\mathbb{R}^d)), \\ \tilde{\rho}(t,\mathbb{R}^d) = \|\gamma^2\|_{L^1} \qquad \textit{for all } t \geq 0, \end{aligned}$$

and there exists $C_0 > 0$ such that

$$\sup_{t\geq 0} \operatorname{ess} \frac{1}{\tau(t)^2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\eta|^2 \,\tilde{W}(t,\mathrm{d}y,\mathrm{d}\eta) + \int_0^\infty \frac{\dot{\tau}(t)}{\tau^3(t)} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\eta|^2 \,\tilde{W}(t,\mathrm{d}y,\mathrm{d}\eta)\,\mathrm{d}t \leq C_0, \tag{1.1.13}$$

$$\int_{\mathbb{R}^d} y\,\tilde{\rho}(t,y)\,\mathrm{d}y = \frac{1}{\tau(t)}(C_1 t + C_2), \qquad \forall t \geq 0, \tag{1.1.14}$$

where

$$C_j = \lim_{n\to\infty} \frac{\|\gamma^2\|_{L^1}}{\|u_{\varepsilon_n,\mathrm{in}}\|_{L^2}^2} I_{j,0}^{\varepsilon_n} \qquad \text{for } j=1,2,$$

which yields

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \end{pmatrix} \tilde\rho(t,y)\,\mathrm{d}y \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \end{pmatrix} \gamma^2(y)\,\mathrm{d}y.$$

Lastly, there exists $C_3 > 0$ such that for all $t \geq 2$,

$$\mathcal{W}_1\left(\frac{\tilde\rho(t)}{\pi^{\frac{d}{2}}}, \frac{\gamma^2}{\pi^{\frac{d}{2}}}\right) \leq \frac{C_3}{\sqrt{\ln t}}. \tag{1.1.15}$$

The main result of this theorem is the fact that the two limits (semiclassical limit and large time behavior) *commute*. This is a strong feature which is rather unusual for those two kinds of limits. Indeed, it is known that such limits do not commute for linear Schrödinger equations with potential, in the context of scattering, with asymptotic states under the form of either WKB (see [157, 159]), or coherent states (see e.g. [51, 92, 90]). In [28], a similar lack of commutativity is proven in the case of the Schrödinger equation with a potential and a cubic nonlinearity.

Remark 1.1.11. Even if we do not have any pointwise convergence for W_ε and if $W(t)$ is defined only for *almost all* $t \in (0,\infty)$ to be a non-negative measure, $\rho(t)$ can be defined for *all* $t \in (0,\infty)$ and is not only a non-negative measure but also an L^1 function. Moreover, we do have continuity in time for $\rho(t)$ with values in $\mathcal{P}_1$ endowed with the Wasserstein metric $\mathcal{W}_1$. Actually, the proof shows that we also get locally *uniform* convergence in time of $|v_{\varepsilon_n}|^2$ to $\rho(t)$ in $\mathcal{P}_1$.

Remark 1.1.12. The convergence for the second momentum stated in (1.1.9) is uniform in ε. Yet, we still cannot conclude for the case "$\varepsilon = 0$" because we do not know if $\int_{\mathbb{R}^d} |y|^2\,\tilde\rho_\varepsilon(t,y)\,\mathrm{d}y$ converges to $\int_{\mathbb{R}^d} |y|^2\,\tilde\rho(t,y)\,\mathrm{d}y$. This would have been the case if, for example, we had an estimate for a higher momentum.

Remark 1.1.13. Remember that if (A2) is satisfied, $I_{i,0}^\varepsilon$ $(i=1,2)$ is independent of ε, therefore C_j $(j=1,2)$ are still the same quantities. Moreover, in the same case, it is known (see [110, Exemple III.5.]) that

$$W_\varepsilon(0) \underset{n\to\infty}{\rightharpoonup} \rho_{\mathrm{in}}(x)\,\mathrm{d}x \otimes \delta_{\xi=\nabla\phi_{\mathrm{in}}(x)} \qquad \text{and} \qquad \tilde W_\varepsilon(0) \underset{n\to\infty}{\rightharpoonup} \frac{\|\gamma^2\|_{L^1}}{\|\rho_{\mathrm{in}}\|_{L^1}} \rho_{\mathrm{in}}(x)\,\mathrm{d}x \otimes \delta_{\xi=\nabla\phi_{\mathrm{in}}(x)} \qquad \text{in } \mathcal{A}'_{w-*}.$$

In the same way as for Corollary 1.1.8, we also have a convergence rate for some other metrics.

Corollary 1.1.14. *With the notations of Theorem 1.1.10, there exists $C_4 > 0$ such that for all $t \geq 2$ and $\delta \in (0,1)$,*

$$\left\|\tilde\rho(t) - \gamma^2\right\|_{W^{-1+\delta,1}} \leq \frac{C_4}{(\ln t)^{\frac{1-\delta}{2}}}, \qquad \mathcal{W}_{1+\delta}\left(\frac{\tilde\rho(t)}{\pi^{\frac{d}{2}}}, \frac{\gamma^2}{\pi^{\frac{d}{2}}}\right) \leq \frac{C_4}{(\ln t)^{\frac{1-\delta}{2}}}.$$

Kinetic Isothermal Euler system. In view of the previous remarks on the Wigner Transform, the Wigner Measure usually satisfies the related kinetic (or Vlasov-type) equation with the same potential, as soon as the potential is smooth enough. Therefore, there is a formal link between the Wigner Measure we found in Theorem 1.1.10 to the kinetic/Vlasov-type equation with the same potential, i.e.:

$$\partial_t f + \xi\cdot\nabla_x f - \lambda\,\nabla_x(\ln\rho)\cdot\nabla_\xi f = 0, \qquad f(0,x,\xi) = f_{\mathrm{in}}(x,\xi), \qquad t>0, (x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d, \tag{1.1.16}$$

where $\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,\mathrm{d}\xi)$. First of all, this equation has a strong link with the *isothermal Euler system*: a time-dependent mono-kinetic measure $f(t,x,\xi) = \rho(t,x)\,\mathrm{d}x \otimes \delta_{\xi=v(t,x)}$ satisfies (1.1.16) if and only if (ρ, v) satisfies:

$$\begin{cases} \partial_t\rho + \nabla_x\cdot(\rho v) = 0, \\ \partial_t(\rho v) + \nabla_x\cdot(\rho v\otimes v) + \lambda\nabla_x\rho = 0. \end{cases} \tag{1.1.17}$$

This is why (1.1.16) is called the *Kinetic Isothermal Euler System* (KIE). Such an equation has already been studied in other contexts, mostly because it arises as the formal *quasineutral limit* of the Vlasov-Poisson system with *massless electrons*, but to the best of our knowledge the studies proving rigorously this quasineutral limit stick to the torus in space (see for instance [93, 80]). Even if it does not apply to our case, another interesting result is worth mentioning: the local well-posedness in 1D for mono-kinetic solutions far from vacuum and whose parameters (ρ, v) are in Zhidkov space with enough regularity (see Theorem 1.4. of [36]).

For our case where the solution should have (at least) the same properties as in Theorem 1.1.10, some results were already found. In particular, R. Carles and A. Nouri proved that the Wigner Transform of solutions to (1.1.1) in 1D with Gaussian initial data converges (even *pointwise in time*) to a monokinetic measure, with ρ Gaussian and v affine in space, solution to (1.1.16) (see [36, Theorem 1.1.]). We will name those solutions to (1.1.16) *Gaussian-monokinetic* solutions. Such a remark strengthens the intuition of a link between (1.1.1) and (1.1.16) through the Wigner Transform.

Actually, even if it is not our purpose to develop a full Cauchy theory in this case, a nice framework for (1.1.16) should give the usual properties for Vlasov-type equations, and such properties are enough to prove the same large time behavior in Wasserstein distance as in Theorems 1.1.5 and 1.1.10 (see Section 1.5.1). This discussion is even more enlightened by the following result, providing a new class of explicit global *strong* solutions to (1.1.16) in 1D: *Gaussian-Gaussian* solutions.

Theorem 1.1.15. *1. For $c_{1,0} > 0$, $c_{2,0} > 0$ and $c_{1,1}, B_0, B_1 \in \mathbb{R}$, define $c_1 \in \mathcal{C}^\infty(\mathbb{R}^+)$ the solution of the ordinary differential equation*

$$\ddot{c}_1 = \frac{2\lambda}{c_1} + \frac{\tilde{C}^2}{c_1^3}, \qquad \tilde{C} := c_{1,0}\, c_{2,0}, \qquad c_1(0) = c_{1,0}, \qquad \dot{c}_1(0) = c_{1,1}. \tag{1.1.18}$$

Then, set

$$c_2(t) := \frac{\tilde{C}}{c_1(t)}, \qquad b_1(t) := B_1 t + B_0, \qquad b_2(t,x) := \frac{\dot{c}_1(t)}{c_1(t)}(x - B_1\, t - B_0) + B_1. \tag{1.1.19}$$

The function $f = f(t,x,\xi)$ defined by

$$f(t,x,\xi) = \frac{1}{\pi\, c_1(t)\, c_2(t)}\ \exp\left[-\frac{|x-b_1(t)|^2}{c_1(t)^2} - \frac{|\xi - b_2(t,x)|^2}{c_2(t)^2}\right] \tag{1.1.20}$$

is a solution to (1.1.16)*. Moreover, if we rescale it to $\tilde{f} = \tilde{f}(t,y,\eta)$ by setting*

$$f(t,x,\xi) := \frac{1}{\|\gamma^2\|_{L^1}}\, \tilde{f}\left(t, \frac{x}{\tau(t)}, \tau(t)\,\xi - \dot{\tau}(t)\, x\right),$$

and define

$$\tilde{\rho}(t,y) := \int_{\mathbb{R}^d} \tilde{f}(t,y,\eta)\, \mathrm{d}\eta,$$

there holds

$$\|\tilde{\rho}(t,.) - \gamma^2\|_{L^1} = \mathrm{O}\left(\sqrt{\frac{\ln\ln t}{\ln t}}\right). \tag{1.1.21}$$

2. *Let $T \in (0,+\infty]$, $b_1 = b_1(t) \in \mathcal{C}^1([0,T),\mathbb{R})$, $c_1 = c_1(t) \in \mathcal{C}^1([0,T),(0,\infty))$, $b_2 = b_2(t,x) \in \mathcal{C}^1([0,T)\times\mathbb{R},\mathbb{R})$ and $c_2 = c_2(t,x) \in \mathcal{C}^1([0,T)\times\mathbb{R},(0,\infty))$ such that*

$$f(t,x,\xi) = \frac{1}{\pi\, c_1(t)\, c_2(t,x)}\ \exp\left[-\frac{|x-b_1(t)|^2}{c_1(t)^2} - \frac{|\xi - b_2(t,x)|^2}{c_2(t,x)^2}\right], \qquad t \in [0,T),\ x,\xi \in \mathbb{R},$$

is a solution to (1.1.16)*. Then c_2 does not depend on x, all the functions are $\mathcal{C}^\infty$ and* (1.1.18)-(1.1.19) *hold.*

Remark 1.1.16. This theorem may also handle the case when $c_{2,0} = 0$, which is actually the monokinetic case where we have a Dirac in ξ:

$$f_{\text{in}}(x,\xi) = \frac{1}{\sqrt{\pi}\, c_{1,0}}\, \exp\left[-\frac{(x-b_{1,0})^2}{c_{1,0}^2}\right] \otimes \delta_{\xi = b_{2,0}(x)}.$$

where $b_{2,0}(x)$ is affine. Then the previous theorem shows that f is a Dirac in ξ for all time (if we only consider Gaussian solutions), as $c_1(t)\, c_2(t) = c_1(0)\, c_2(0) = 0$ with $c_1(t) \neq 0$ for all $t > 0$. This is similar to [36].

Remark 1.1.17. We stated this result in 1D, however we can extend this class of solutions (and also the Gaussian-monokinetic class) to any dimension d by tensor product. Indeed, in the same way as for (1.1.1) (see [23]), the tensor product of two solutions to (1.1.16) is still a solution to (1.1.16).

Remark 1.1.18. It is worth noting that the expected large time behavior still holds, due to the fact that the behavior of c_1 has already been studied in [36]:

$$c_1(t) \underset{t\to\infty}{\sim} 2t\sqrt{\lambda \ln t},$$

with the better result of strong convergence of $\tilde{\rho}$ to γ^2 in L^1 with a slightly slower convergence. This does not mean that the convergence in Wasserstein distance is slower for this class of solutions. Actually, we can prove that the convergence rate found in Theorem 1.1.10 still holds in this case, despite the fact that those solutions do not fit with any Wigner Measure.

1.1.4. Outline of the paper

In Section 1.2 we review and extend some of the standard facts on the Wigner Transform. Section 1.3 is devoted to the study of the semiclassical limit of (1.1.1) through the first part of the proof of Theorems 1.1.5 and 1.1.10, which is everything except the convergence rate in Wasserstein distance: we extend the results of [33] to (1.1.1) (with semiclassical constant) and then use the results on the Wigner Transform to perform the semiclassical limit. A sharpened analysis of the Fokker-Planck equation (which already gives the weak convergence in [33]) is provided in Section 1.4. The estimates coming from this analysis lead to the convergence rate in Wasserstein metric. Finally, Section 1.5 is split into two parts. The first part contains a discussion of the Kinetic Isothermal Euler system and its (formal) properties. We show that those properties are enough to get the same behavior as in Theorem 1.1.10, through an intermediate result we prove in Section 1.4. The last part deals with Theorem 1.1.15 and its new class of explicit solutions to (1.1.16).

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1.2. Wigner Transform and Wigner Measure

This section is devoted to the Wigner Transform and Wigner Measure. Even if they have already been studied a lot (see [110, 10, 88, 87]), many standard facts about them were proved without taking into account the time dependence. Indeed, the further results of the convergence of the Wigner Measure of a solution to a Schrödinger equation to the related kinetic/Vlasov-type equation conclude to a convergence which is pointwise in time for a lot of cases (see for instance [110, Théorèmes IV.1. and IV.2.]), therefore those facts are enough to get suitable properties for the Wigner Measure. However, those results do not fall within our framework, so we need to extend those basic facts to the case with time dependence.

1.2.1. Definitions and first property

For any sequence of functions $f_\varepsilon = f_\varepsilon(x) \in L^2(\mathbb{R}^d)$ for $\varepsilon > 0$, define the Wigner Transform W_ε by

$$W_\varepsilon(x,\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi\cdot z} f_\varepsilon\left(x + \frac{\varepsilon z}{2}\right) \overline{f_\varepsilon\left(x - \frac{\varepsilon z}{2}\right)} \,\mathrm{d}z = \mathcal{F}_z \tilde{\rho}_\varepsilon(x,\xi), \qquad (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d,$$

where

$$\tilde{\rho}_\varepsilon(x,z) = f_\varepsilon\left(x + \frac{\varepsilon z}{2}\right) \overline{f_\varepsilon\left(x - \frac{\varepsilon z}{2}\right)}, \qquad (x,z) \in \mathbb{R}^d \times \mathbb{R}^d.$$

W_ε is a real-valued function on the phase space. However, it may be non-integrable and sometimes negative. Both issues are fixed by working with the Husimi Transform, which is a slight modification of the Wigner Transform. For this purpose, we define the Gaussian with ε variance:

$$\gamma_\varepsilon(x) = \frac{1}{(\pi\varepsilon)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{\varepsilon}\right), \qquad G_\varepsilon(x,\xi) = \gamma_\varepsilon(x)\,\gamma_\varepsilon(\xi), \qquad \text{for } x,\xi \in \mathbb{R}^d.$$

This leads to the definition of the Husimi Transform W_ε^H:

$$W_\varepsilon^H = W_\varepsilon * G_\varepsilon = W_\varepsilon *_x \gamma_\varepsilon *_\xi \gamma_\varepsilon. \tag{1.2.1}$$

The fact that the Husimi Transform is non-negative and integrable is not obvious at first sight, but this is well-known (see [110]).

Proposition 1.2.1. *The Husimi Transform $W_\varepsilon^H = W_\varepsilon^H(x,\xi)$ of any function $f_\varepsilon \in L^2(\mathbb{R}^d)$ defined by* (1.2.1) *is non-negative and satisfies*

$$\int_{\mathbb{R}^d} W_\varepsilon^H(x,\xi)\,\mathrm{d}\xi = |f_\varepsilon|^2 * \gamma_\varepsilon(x), \qquad \textit{for all } x \in \mathbb{R}^d \tag{1.2.2}$$

and

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} W_\varepsilon^H(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x = \|f_\varepsilon\|_{L^2}^2.$$

1.2.2. Momenta

The fact that the Husimi Transform is non-negative is very useful in order to compute some momenta. As it is a slight modification of the Wigner Transform, their computation leads to some interesting estimates even in the limit $\varepsilon \to 0$.

Proposition 1.2.2. *Given any $f_\varepsilon \in L^2(\mathbb{R}^d)$, $\varepsilon > 0$ and its Husimi Transform W_ε^H, there holds for all $x \in \mathbb{R}^d$:*

1. *If $f_\varepsilon \in H^1(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} |\xi|^2\, W_\varepsilon^H(x,\xi)\,\mathrm{d}\xi = \varepsilon^2\, |\nabla f_\varepsilon|^2 * \gamma_\varepsilon(x) - \frac{\varepsilon^2}{4}\, |f_\varepsilon|^2 * \Delta\gamma_\varepsilon(x) + \frac{\varepsilon d}{2} |f_\varepsilon|^2 * \gamma_\varepsilon(x), \tag{1.2.3}$$

and

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} |\xi|^2\, W_\varepsilon^H(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x = \varepsilon^2 \|\nabla f_\varepsilon\|_{L^2}^2 + \frac{\varepsilon d}{2}\|f_\varepsilon\|_{L^2}^2. \tag{1.2.4}$$

In a more general way,

$$\int_{\mathbb{R}^d} \xi_i\xi_j\, W_\varepsilon^H(x,\xi)\,\mathrm{d}\xi = \varepsilon^2\ \mathrm{Re}\left(\partial_i f_\varepsilon\, \overline{\partial_j f_\varepsilon}\right) * \gamma_\varepsilon(x) - \frac{\varepsilon^2}{4}\, |f_\varepsilon|^2 * \partial_i\partial_j\gamma_\varepsilon(x) + \frac{\varepsilon\,\delta_{ij}}{2} |f_\varepsilon|^2 * \gamma_\varepsilon(x), \tag{1.2.5}$$

and

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \xi_i\xi_j\, W_\varepsilon^H(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x = \varepsilon^2 \int_{\mathbb{R}^d} \mathrm{Re}\left(\partial_i f_\varepsilon\, \overline{\partial_j f_\varepsilon}\right)\mathrm{d}x + \frac{\varepsilon\,\delta_{ij}}{2}\|u_\varepsilon\|_{L^2}^2. \tag{1.2.6}$$

2. *If $f_\varepsilon \in H^1(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} \xi\, W_\varepsilon^H(x,\xi)\,\mathrm{d}\xi = \varepsilon\,\mathrm{Im}(\nabla f_\varepsilon\, \overline{f_\varepsilon}) * \gamma_\varepsilon\,(x)\,, \tag{1.2.7}$$

and therefore

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \xi\, W_\varepsilon^H(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x = \int_{\mathbb{R}^d} \varepsilon\,\mathrm{Im}(\nabla f_\varepsilon\, \overline{f_\varepsilon})\,\mathrm{d}x. \tag{1.2.8}$$

3. *If $u_\varepsilon \in \mathcal{F}(H^1)$,*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x|^2\, W_\varepsilon^H(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi = \|x\, f_\varepsilon\|_{L^2}^2 + \frac{\varepsilon d}{2}\,\|f_\varepsilon\|_{L^2}^2. \tag{1.2.9}$$

The proof is very computational and will be done in Appendix 1.A.

1.2.3. Semiclassical limit

Even if the Wigner Transform is not integrable, we still have some bounds thanks to the following Banach space (and algebra) of test functions:

$$\mathcal{A} = \left\{\phi \in \mathcal{C}_0(\mathbb{R}_x^d \times \mathbb{R}_\xi^d), (\mathcal{F}_\xi\phi)(x,z) \in L^1(\mathbb{R}_z^d, \mathcal{C}_0(\mathbb{R}_x^d))\right\}$$

endowed with the norm

$$\|\phi\|_{\mathcal{A}} = \|\mathcal{F}_\xi\phi\|_{L_z^1 L_x^\infty}.$$

It is known that, for any sequence $(f_\varepsilon = f_\varepsilon(x))_{\varepsilon>0}$ bounded in $L^2(\mathbb{R}^d)$, its Wigner Transform W_ε is uniformly bounded in $\mathcal{A}'$ and therefore weak-$*$ converges (up to the extraction of a subsequence) to a (non-negative) measure, called a Wigner Measure (see [110, Proposition III.1.]). Adding the time-dependence to the boundedness is obviously easy. However, reaching the limit with the addition of the time-dependence is a bit more difficult. Usually, the (Schrödinger) equation satisfied by u_ε yields an equation on W_ε from which one can derive some equicontinuity if the potential is smooth enough, but here the potential is highly singular because we do not have any control near the vacuum. Yet, the uniform bound in $L^\infty\left((0,T),\mathcal{A}'\right)$ implies the uniform bound in $L^{p'}\left((0,T),\mathcal{A}'\right) = \left(L^p\left((0,T),\mathcal{A}\right)\right)'$ for any $p > 1$. This remark shows that we can extend the result of [110, Théorème III.1.] with time-dependence, even if it means losing pointwise convergence in time.

We say that a sequence $(f_\varepsilon)_{\varepsilon>0}$ of functions $f_\varepsilon = f_\varepsilon(t,x) \in L^\infty\left((0,T), H^1\cap\mathcal{F}(H^1)(\mathbb{R}^d)\right)$ satisfies the assumption (A3) for some $T > 0$ if

$$f_\varepsilon,\ x\,f_\varepsilon \text{ and } \varepsilon\nabla f_\varepsilon \text{ are uniformly bounded in } L^\infty\left((0,T), L^2(\mathbb{R}^d)\right). \tag{A3}$$

Lemma 1.2.3. *1. Given any sequence $(f_\varepsilon)_{\varepsilon>0}$ of functions $f_\varepsilon = f_\varepsilon(t,x) \in L^\infty\left((0,T), L^2(\mathbb{R}^d)\right)$ uniformly bounded, there exists a subsequence $(\varepsilon_n)_n$ such that $\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$ and there exists a (non-negative) measure W (called Wigner Measure) in $L^\infty((0,T),\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d))$ such that for every $p \in (1,\infty)$*

$$W_{\varepsilon_n} \underset{n\to\infty}{\rightharpoonup} W \qquad \text{in } L^p\left((0,T),\mathcal{A}'_{w-*}\right), \qquad\qquad W_{\varepsilon_n}^H \underset{n\to\infty}{\rightharpoonup} W \qquad \text{in } L^p\left((0,T),\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d)\right).$$

2. *Moreover, if* $f_\varepsilon = f_\varepsilon(t,x)$ *satisfy* (A3)*, then* $W^H_{\varepsilon_n} \underset{n\to\infty}{\longrightarrow} W$ *in* $L^p\left((0,T),\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d)\right)$ *and the following properties hold for a.e.* $t\in(0,T)$ *and all* $p>1$*,*
   - *On the second momentum in* x*:*
   $$|x|^2\, W^H_{\varepsilon_n} \underset{n\to\infty}{\rightharpoonup} |x|^2\, W \quad \text{in } L^p\left((0,T),\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d)\right), \qquad \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x|^2\, W(t,\mathrm{d}x,\mathrm{d}\xi) \le \liminf_{n\to\infty} \|x f_{\varepsilon_n}\|^2_{L^\infty_t L^2_x},$$
   - *On the second momentum in* ξ*:*
   $$|\xi|^2\, W^H_{\varepsilon_n} \underset{n\to\infty}{\rightharpoonup} |\xi|^2\, W \quad \text{in } L^p\left((0,T),\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d)\right), \qquad \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\xi|^2\, W(t,\mathrm{d}x,\mathrm{d}\xi) \le \liminf_{n\to\infty} (\varepsilon_n)^2\|\nabla f_{\varepsilon_n}\|^2_{L^\infty_t L^2_x},$$
   - *On the density:*
   $$|f_{\varepsilon_n}|^2 \underset{n\to\infty}{\longrightarrow} \rho := \int_{\mathbb{R}^d} W(.,.,\mathrm{d}\xi) \quad \text{in } L^p\left((0,T),\mathcal{M}(\mathbb{R}^d)\right), \qquad \|f_{\varepsilon_n}\|^2_{L^2} \underset{n\to\infty}{\rightharpoonup} \rho(.,\mathbb{R}^d) \quad \text{in } L^p((0,T)).$$

Remark 1.2.4. $W_{\varepsilon_n} \underset{n\to\infty}{\rightharpoonup} W$ in $L^p((0,T),\mathcal{A}'_{w-*})$ means that for any $\phi=\phi(t,x,\xi)$ such that $\|\phi(t)\|_{\mathcal{A}} \in L^{p'}((0,T))$, $\int_0^T\iint_{\mathbb{R}^d\times\mathbb{R}^d}\phi(t,x,\xi)\,W_{\varepsilon_n}(t,x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}t$ converges to $\int_0^T\iint_{\mathbb{R}^d\times\mathbb{R}^d}\phi(t,x,\xi)\,W(t,\mathrm{d}x,\mathrm{d}\xi)\,\mathrm{d}t$. In the same way, $W^H_{\varepsilon_n} \underset{n\to\infty}{\rightharpoonup} W$ in $L^p((0,T),\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d))$ means that for any $\phi=\phi(t,x,\xi)$ such that $\phi(t)\in\mathcal{C}_0(\mathbb{R}^d\times\mathbb{R}^d)$ (continuous and going to 0 at infinity) for a.e. $t\in(0,T)$ and $\|\phi(t)\|_{L^\infty}\in L^{p'}(0,T)$, $\int_0^T\iint_{\mathbb{R}^d\times\mathbb{R}^d}\phi(t,x,\xi)\,W^H_{\varepsilon_n}(t,x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}t$ converges to $\int_0^T\iint_{\mathbb{R}^d\times\mathbb{R}^d}\phi(t,x,\xi)\,W(t,\mathrm{d}x,\mathrm{d}\xi)\,\mathrm{d}t$. The same kind of remark holds for the convergence $|f_{\varepsilon_n}|^2 \underset{n\to\infty}{\longrightarrow} \rho$ when taking $\phi\in L^{p'}\left((0,T),\mathcal{C}_b(\mathbb{R}^d)\right)$, and also for $W^H_{\varepsilon_n} \underset{n\to\infty}{\longrightarrow} W$.

Remark 1.2.5. The assumption (A3) is not the sharpest for the results about the density ρ. Actually, one shall only need some *ε-oscillatory* and *compact at infinity* feature *uniformly in time* for the sequence $(f_\varepsilon)_{\varepsilon>0}$. However, the assumption (A3) makes the proof easier, also allows us to get good properties on the second momentum of the Wigner Measure and is actually sufficient for our further results.

Proof. The first part of the proof is actually a re-writing of the proof of the first part of [110, Théorème III.1.], with in addition the time-dependence. W_ε and W^H_ε are bounded respectively in $L^\infty((0,T),\mathcal{A}')$ and in $L^\infty\left((0,T),L^1(\mathbb{R}^d\times\mathbb{R}^d)\right)$. Thus, there exists a subsequence ε_n such that W_{ε_n} (resp. $W^H_{\varepsilon_n}$) weakly converges in $L^p\left((0,T),\mathcal{A}'_{w-*}\right)$ (resp. $L^p\left((0,T),\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d)\right)$) for all $p\in(1,\infty)$ to a limit $W\in L^\infty((0,T),\mathcal{A}')$ (resp. $W^T_H\in L^\infty\left((0,T),\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d)\right)$). Following the idea of [110, Théorème III.1.], we should be able to prove that $W=W_H$. Since we have

$$W^H_\varepsilon = W_\varepsilon * G_\varepsilon, \qquad \text{where } G_\varepsilon = \frac{1}{(\pi\varepsilon)^d}\,e^{-\frac{|x|^2+|\xi|^2}{\varepsilon}},$$

it is enough to prove that, for example, for any $\phi\in\mathcal{C}^\infty_c((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ which is a dense subset of $L^2((0,T),\mathcal{A})$, $\phi*G_\varepsilon$ converges in $L^2((0,T),\mathcal{A})$ to ϕ. Knowing that

$$\mathcal{F}_\xi(\phi*G_\varepsilon)(t,x,z) = \left[\mathcal{F}_\xi\phi(t,x,z) *_x \frac{1}{(\pi\varepsilon)^{\frac d2}}\,e^{-\frac{|x|^2}{\varepsilon}}\right] e^{-\varepsilon\frac{|z|^2}{4}},$$

we see that,

$$\|\phi(t)*G_\varepsilon-\phi(t)\|_{\mathcal{A}} \le \int_{\mathbb{R}^d}\sup_x\left|\mathcal{F}_\xi\phi(t)-\mathcal{F}_\xi\phi(t) *_x \frac{1}{(\pi\varepsilon)^{\frac d2}}\,e^{-\frac{|x|^2}{\varepsilon}}\right|\mathrm{d}z + \int_{\mathbb{R}^d}(1-e^{-\varepsilon\frac{|z|^2}{4}})\sup_x|\mathcal{F}_\xi\phi(t)|\,\mathrm{d}z.$$

The second term goes to 0 when ε goes to 0 by dominated convergence for all $t\in(0,T)$, and so does the first term since $\mathcal{F}_\xi\phi(t)\in\mathcal{S}(\mathbb{R}^d\times\mathbb{R}^d)$. Moreover,

$$\|\mathcal{F}_\xi(\phi*G_\varepsilon)(t)\|_{L^1_zL^\infty_x} = \left\|\left[\mathcal{F}_\xi\phi(t) *_x \frac{1}{(\pi\varepsilon)^{\frac d2}}e^{-\frac{|x|^2}{\varepsilon}}\right]e^{-\varepsilon\frac{|z|^2}{4}}\right\|_{L^1_zL^\infty_x} \le \|\mathcal{F}_\xi\phi(t)\|_{L^1_zL^\infty_x} = \|\phi(t)\|_{\mathcal{A}},$$

which yields

$$\|\phi(t)*G_\varepsilon-\phi(t)\|_{\mathcal{A}} \le \|\phi(t)*G_\varepsilon\|_{\mathcal{A}} + \|\phi(t)\|_{\mathcal{A}} \le 2\|\phi(t)\|_{\mathcal{A}}.$$

Then, we conclude by dominated convergence

$$\int_0^T \|\phi(t) * G_\varepsilon - \phi(t)\|_{\mathcal{A}}^2 \,\mathrm{d}t \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

which is what we wanted. Therefore, $W = W^H$.

The proof of part 2 is rather usual. First, take some non-increasing $\chi \in \mathcal{C}_c^\infty([0,\infty))$ such that $\chi \equiv 1$ on $[0,1]$ and $0 \leq \chi \leq 1$. Given any non-negative function $\phi = \phi(t) \in \mathcal{C}_c^\infty((0,T))$, we know that for any $\delta > 0$,

$$\int_0^T \iint \phi(t)\,\chi(\delta|x|^2)\,|x|^2\, W_{\varepsilon_n}^H(t,x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}t \underset{n\to\infty}{\longrightarrow} \int_0^T \iint \phi(t)\,\chi(\delta|x|^2)\,|x|^2\, W(t,\mathrm{d}x,\mathrm{d}\xi)\,\mathrm{d}t,$$

Since all the factors are non-negative, the term on the left-hand side is bounded thanks to (1.2.9):

$$\begin{aligned}\int_0^T \iint \phi(t)\,\chi(\delta|x|^2)\,|x|^2\, W_{\varepsilon_n}^H(t,x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}t &\leq \int_0^T \phi(t) \iint |x|^2\, W_{\varepsilon_n}^H(t,x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}t \\ &\leq \int_0^T \phi(t)\left[\|x\, f_{\varepsilon_n}\|_{L^2}^2 + \frac{\varepsilon_n d}{2}\,\|f_{\varepsilon_n}\|_{L^2}^2\right]\mathrm{d}t \\ &\leq \|\phi\|_{L^1}\left[\|x\, f_{\varepsilon_n}\|_{L_t^\infty L_x^2}^2 + \frac{\varepsilon_n d}{2}\,\|f_{\varepsilon_n}\|_{L_t^\infty L_x^2}^2\right].\end{aligned}$$

Therefore, we get

$$\int_0^T \iint \phi(t)\,\chi(\delta|x|^2)\,|x|^2\, W(t,\mathrm{d}x,\mathrm{d}\xi)\,\mathrm{d}t \leq \|\phi\|_{L^1} \liminf_{n\to\infty} \|x f_{\varepsilon_n}\|_{L_t^\infty L_x^2}^2,$$

and we conclude thanks to the monotone convergence theorem as $\delta \to 0$:

$$\int_0^T \iint \phi(t)\,|x|^2\, W(t,\mathrm{d}x,\mathrm{d}\xi)\,\mathrm{d}t \leq \|\phi\|_{L^1} \liminf_{n\to\infty} \|x f_{\varepsilon_n}\|_{L_t^\infty L_x^2}^2,$$

hence

$$\iint |x|^2\, W(t,\mathrm{d}x,\mathrm{d}\xi) \leq \liminf_{n\to\infty} \|x f_{\varepsilon_n}\|_{L_t^\infty L_x^2}^2 \qquad \text{for a.e. } t \in (0,T),$$

and the same proof holds for the second momentum in ξ thanks to (1.2.4). Getting this second momentum leads to the following result with a usual argument:

$$|x|^2\, W_{\varepsilon_n}^H \underset{n\to\infty}{\rightharpoonup} |x|^2\, W \quad \text{in } L^p((0,T),\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d)), \qquad |\xi|^2\, W_{\varepsilon_n}^H \underset{n\to\infty}{\rightharpoonup} |\xi|^2\, W \quad \text{in } L^p((0,T),\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d)),$$

and thus

$$W_{\varepsilon_n}^H \underset{n\to\infty}{\longrightarrow} W \qquad \text{in } L^p((0,T),\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d)).$$

In particular, thanks to (1.2.2),

$$\int_{\mathbb{R}^d} W_{\varepsilon_n}^H(t,x,\xi)\,\mathrm{d}\xi = |f_{\varepsilon_n}(t,.)|^2 * \gamma_{\varepsilon_n}(x) \underset{n\to\infty}{\longrightarrow} \rho \qquad \text{in } L^p((0,T),\mathcal{M}(\mathbb{R}^d)).$$

But $(f_{\varepsilon_n})_n$ is uniformly bounded in $L^\infty\left((0,T),\mathcal{F}(H^1)(\mathbb{R}^d)\right)$, thus we get up to a further subsequence

$$|f_{\varepsilon_n}|^2 \underset{n\to\infty}{\longrightarrow} \tilde{\rho} \qquad \text{in } L^p((0,T),\mathcal{M}(\mathbb{R}^d)),$$

for some $\tilde{\rho} \in L^\infty((0,T),\mathcal{M}(\mathbb{R}^d))$. In particular, it is also known that

$$|f_{\varepsilon_n}(t,.)|^2 * \gamma_{\varepsilon_n}(x) \underset{n\to\infty}{\longrightarrow} \tilde{\rho} \qquad \text{in } L^p((0,T),\mathcal{M}(\mathbb{R}^d)).$$

Therefore $\tilde{\rho} = \rho$, hence the whole sequence $(|f_{\varepsilon_n}|^2)_n$ converges (there is no need of further subsequence) and especially

$$\|f_{\varepsilon_n}(t)\|_{L^2}^2 = \int_{\mathbb{R}^d} |f_{\varepsilon_n}(t,x)|^2\,\mathrm{d}x \underset{n\to\infty}{\rightharpoonup} \int_{\mathbb{R}^d} \tilde{\rho}(t,\mathrm{d}x) \qquad \text{in } L^p((0,T)).$$ □

1.3. Semiclassical limit of the Logarithmic Schrödinger equation

In this section, we prove Theorems 1.1.4, 1.1.5 and 1.1.10 except the convergence rates in Wasserstein distance, which will be done in the next section. First, a brief proof of Theorem 1.1.4 and a longer one for Theorem 1.1.5 are given. Using those results along with the properties of Section 1.2, the semiclassical limit is then performed and gives the first part of the proof of the latter.

1.3.1. Proof of theorems 1.1.4 and 1.1.5

The proof of Theorem 1.1.4 is very easy and follows from a simple change of variable: u_ε is a solution to (1.1.1) if and only if $\tilde{u}_\varepsilon(t,x) = u_\varepsilon(\varepsilon t, \varepsilon x)$ is solution to (1.1.2) (with initial data $u_{\varepsilon,\text{in}}(\varepsilon x)$). Therefore, we can use [33, Theorem 1.5.] and it leads to the conclusion with some additional and obvious computations. For Theorem 1.1.5, the first part of the proof is actually a slight and simple adaptation of the proof of [33, Theorem 1.7.].

Rescaling and estimates. Writing (1.1.1) in terms of v_ε yields

$$i\varepsilon\,\partial_t v_\varepsilon + \frac{\varepsilon^2}{2\tau(t)^2}\Delta_y v_\varepsilon = \lambda v_\varepsilon \ln\left|\frac{v_\varepsilon}{\gamma}\right|^2 - \lambda\left(d\ln\tau(t) - 2\,\ln\frac{\|u_{\varepsilon,\text{in}}\|_{L^2}}{\|\gamma\|_{L^2}}\right)v_\varepsilon.$$

The last term is totally harmless, as it can be removed by changing v_ε into $v_\varepsilon\, e^{-i\frac{\theta}{\varepsilon}}$ where

$$\theta = \theta(t) := \lambda d\int_0^t \ln\tau(s)\,ds - 2\lambda t\ln\frac{\|u_{\varepsilon,\text{in}}\|_{L^2}}{\|\gamma\|_{L^2}}.$$

Thus, we obtain the system

$$i\varepsilon\,\partial_t v_\varepsilon + \frac{\varepsilon^2}{2\tau(t)^2}\Delta_y v_\varepsilon = \lambda v_\varepsilon \ln\left|\frac{v_\varepsilon}{\gamma}\right|^2, \qquad v_\varepsilon(0,x) = \frac{\|\gamma\|_{L^2}}{\|u_{\varepsilon,\text{in}}\|_{L^2}}\,u_{\varepsilon,\text{in}}. \tag{1.3.1}$$

We define the modified total energy and kinetic energy with semiclassical constant and the relative entropy:

$$\mathcal{E}^\varepsilon_{\text{kin}}(t) := \frac{\varepsilon^2}{2\,\tau(t)^2}\|\nabla v_\varepsilon\|^2_{L^2}, \qquad \mathcal{E}^\varepsilon_{\text{ent}}(t) := \int_{\mathbb{R}^d}|v_\varepsilon(t,y)|^2\ln\left|\frac{v_\varepsilon(t,y)}{\gamma(y)}\right|^2\mathrm{d}y,$$
$$\mathcal{E}^\varepsilon := \mathcal{E}^\varepsilon_{\text{kin}} + \lambda\,\mathcal{E}^\varepsilon_{\text{ent}}.$$

Then there holds

$$\dot{\mathcal{E}}^\varepsilon = -2\frac{\dot\tau(t)}{\tau(t)}\mathcal{E}^\varepsilon_{\text{kin}}, \tag{1.3.2}$$

Following the ideas of [33], we should now have estimates which should depend only on $\mathcal{E}^\varepsilon(0)$. However, writing

$$\mathcal{E}^\varepsilon_{\text{ent}}(t) = \int_{\mathbb{R}^d}|v_\varepsilon(t,y)|^2\ln|v_\varepsilon(t,y)|^2\,\mathrm{d}y + \int_{\mathbb{R}^d}|y|^2\,|v_\varepsilon(t,y)|^2\,\mathrm{d}y,$$

it is obvious that $\mathcal{E}^\varepsilon \leq \tilde{E}_\varepsilon(.,v_\varepsilon)$ and in particular $\mathcal{E}^\varepsilon(0) \leq \tilde{E}^0_\varepsilon(v_{\varepsilon,in})$. Actually, if we separate the positive and negative parts of the entropy in the modified total energy thanks to

$$\int|v_\varepsilon|\ln|v_\varepsilon|^2 = \int_{|v_\varepsilon|>1}|v_\varepsilon|^2\ln|v_\varepsilon|^2 + \int_{|v_\varepsilon|\leq 1}|v_\varepsilon|^2\ln|v_\varepsilon|^2,$$

we can define

$$\mathcal{E}^\varepsilon_+ := \mathcal{E}^\varepsilon_{\text{kin}} + \lambda\int_{|v_\varepsilon|>1}|v_\varepsilon|^2\ln|v_\varepsilon|^2 + \lambda\int|y|^2\,|v_\varepsilon|^2 \geq 0,$$
$$\mathcal{E}^\varepsilon_- := -\lambda\int_{|v_\varepsilon|\leq 1}|v_\varepsilon|\ln|v_\varepsilon|^2 \geq 0.$$

Then, with the definition of $\tilde{E}_\varepsilon$ in (1.1.4), it is clear that

$$\tilde{E}_\varepsilon \approx \mathcal{E}^\varepsilon_+ + \mathcal{E}^\varepsilon_- \geq \mathcal{E}^\varepsilon_+ \geq \mathcal{E}^\varepsilon = \mathcal{E}^\varepsilon_+ - \mathcal{E}^\varepsilon_-.$$

We already know that $\mathcal{E}^\varepsilon$ is bounded since it is decreasing and non-negative thanks to the Csiszár-Kullback inequality, which reads (see [7, Theorem 8.2.7])

$$\mathcal{E}^\varepsilon_{\text{ent}}(t) \geq \frac{1}{2\|\gamma^2\|_{L^1(\mathbb{R}^d)}}\left\|\,|v_\varepsilon|^2(t) - \gamma^2\right\|^2_{L^1(\mathbb{R}^d)}.$$

Actually, the following lemma states not only the boundedness of $\tilde{E}_\varepsilon$ but also some integrability property for the $\dot{H}^1$ norm, which are (1.1.6) and (1.1.7).

Lemma 1.3.1. *With the previous notations, there exists a continuous non-decreasing function $C : [0,\infty) \to [0,\infty)$ depending only on λ and d such that for all $t \geq 0$ and for all $\varepsilon > 0$,*

$$\tilde{E}_\varepsilon\big(t, v(t)\big) \leq C\Big(\tilde{E}^0_\varepsilon(v_{\varepsilon,in})\Big), \qquad \int_0^\infty \frac{\varepsilon^2\,\dot{\tau}(t)}{\tau^3(t)} \|\nabla_y v_\varepsilon(t)\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}t \leq C\Big(\tilde{E}^0_\varepsilon(v_{\varepsilon,in})\Big).$$

Proof. Using the fact that the modified energy is non-increasing, we have

$$\mathcal{E}^\varepsilon_+ \leq \mathcal{E}^\varepsilon(0) + \mathcal{E}^\varepsilon_-.$$

The last term can be controlled by

$$\mathcal{E}^\varepsilon_- \leq C_\delta \int_{\mathbb{R}^d} |v_\varepsilon|^{2-\delta},$$

for all $\delta \in (0,2)$. Moreover, we have the estimate

$$\int_{\mathbb{R}^d} |v_\varepsilon|^{2-\delta} \leq C_\delta\, \|v_\varepsilon\|_{L^2}^{2-\left(1+\frac{d}{2}\right)\delta} \|y v_\varepsilon\|_{L^2}^{\frac{d\delta}{2}} = C_\delta\, \|\gamma\|_{L^2}^{2-\left(1+\frac{d}{2}\right)\delta} \|y v_\varepsilon\|_{L^2}^{\frac{d\delta}{2}},$$

as soon as $0 < \delta < \frac{2}{d+2}$ in the same way as (1.1.11). Taking (for example) $\delta = \frac{1}{d+2}$, this implies

$$\mathcal{E}^\varepsilon_- \leq C_d\, (\mathcal{E}^\varepsilon_+)^{\frac{d}{4(d+2)}}, \qquad \text{and then} \qquad \mathcal{E}^\varepsilon_+ \leq \tilde{E}^0_\varepsilon(v_{\varepsilon,in}) + C_d\, (\mathcal{E}^\varepsilon_+)^{\frac{d}{4(d+2)}}.$$

Thus $\mathcal{E}^\varepsilon_+ \leq \tilde{C}\Big(\tilde{E}^0_\varepsilon(v_{\varepsilon,in})\Big)$ for some continuous and non-decreasing function $\tilde{C} : [0,\infty) \to [0,\infty)$ (independent of t and ε) since $\frac{d}{4(d+2)} < 1$. There also holds $\mathcal{E}^\varepsilon_- \leq C_d \left(\tilde{C}\Big(\tilde{E}^0_\varepsilon(v_{\varepsilon,in})\Big)\right)^{\frac{d}{4(d+2)}}$, and then (1.1.6) for $C := \tilde{C} + C_d\, \tilde{C}^{\frac{d}{4(d+2)}}$.

Last, (1.1.7) follows from (1.3.2) and the fact that $\mathcal{E}^\varepsilon(t)$ is bounded uniformly in $t \geq 0$ by $C\left(\tilde{E}^0_\varepsilon(v_{\varepsilon,in})\right)$. □

Remark 1.3.2. The Csiszár-Kullback inequality shows that, if we had $\mathcal{E}^\varepsilon_{\text{ent}}(t) \underset{t\to\infty}{\longrightarrow} 0$ (for example, $\mathcal{E}^\varepsilon(t) \underset{t\to\infty}{\longrightarrow} 0$), we would have $\big\|\, |v_\varepsilon|^2(t) - \gamma^2 \big\|^2_{L^1(\mathbb{R}^d)} \underset{t\to\infty}{\longrightarrow} 0$ and then strong convergence would follow, but we cannot reach this conclusion in the general case.

Convergence of some quadratic quantities. We now prove (1.1.8)-(1.1.9), as stated in the next lemma.

Lemma 1.3.3. *Under the assumptions of Theorem 1.1.5, the first two momenta converge: for all $t \geq 1$ and all $\varepsilon > 0$,*

$$\int_{\mathbb{R}^d} y\,|v_\varepsilon(t,y)|^2\,\mathrm{d}y = \frac{1}{\tau(t)} \frac{\|\gamma^2\|_{L^1}}{\|u_{\varepsilon,\text{in}}\|^2_{L^2}} (I^\varepsilon_{1,0}\, t + I^\varepsilon_{2,0}),$$

$$\left| \int_{\mathbb{R}^d} |y|^2\,|v_\varepsilon(t,y)|^2\,\mathrm{d}y - \int_{\mathbb{R}^d} |y|^2\,\gamma^2(y)\,\mathrm{d}y \right| \leq C\Big(\tilde{E}^0_\varepsilon(v_{\varepsilon,in})\Big)\, \frac{\dot{\tau}(t)+1}{\dot{\tau}(t)^2},$$

where

$$I^\varepsilon_{1,0} = \operatorname{Im}\varepsilon \int_{\mathbb{R}^d} \overline{u_{\varepsilon,\text{in}}}\,\nabla u_{\varepsilon,\text{in}}\,\mathrm{d}y, \qquad I^\varepsilon_{2,0} = \int_{\mathbb{R}^d} y\,|u_{\varepsilon,\text{in}}|^2\,\mathrm{d}y.$$

Proof. Introducing

$$I^\varepsilon_1(t) := \operatorname{Im}\varepsilon \int_{\mathbb{R}^d} \overline{v_\varepsilon}(t,y)\,\nabla v_\varepsilon(t,y)\,\mathrm{d}y, \qquad I^\varepsilon_2(t) := \int_{\mathbb{R}^d} y\,|v_\varepsilon(t,y)|^2\,\mathrm{d}y, \qquad \tilde{I}^\varepsilon_2(t) := \tau(t)\, I^\varepsilon_2(t),$$

we compute

$$\dot{I}^\varepsilon_1 = -2\lambda I^\varepsilon_2, \qquad \dot{I}^\varepsilon_2 = \frac{1}{\tau^2(t)} I^\varepsilon_1, \qquad \ddot{\tilde{I}}^\varepsilon_2 = 0.$$

Therefore (1.1.8) easily follows from simple computations. We now go back to the conservation of energy for u_ε,

$$\frac{\varepsilon^2}{2}\|\nabla u_\varepsilon(t)\|^2_{L^2} + \lambda \int_{\mathbb{R}^d} |u_\varepsilon(t,x)|^2 \ln|u_\varepsilon(t,x)|^2\,\mathrm{d}x = \frac{\varepsilon^2}{2}\,\|\nabla u_{\varepsilon,\text{in}}\|^2_{L^2} + \lambda \int_{\mathbb{R}^d} |u_{\varepsilon,\text{in}}|^2 \ln|u_{\varepsilon,\text{in}}|^2,$$

and translate this property into estimates on v_ε

$$\mathcal{E}_{\text{kin}} + \frac{\dot{\tau}^2}{2} \int |y|^2\,|v_\varepsilon|^2 - \varepsilon\frac{\dot{\tau}}{\tau} \operatorname{Im} \int v_\varepsilon(t,y)\, y\, \overline{\nabla v_\varepsilon}(t,y)\,\mathrm{d}y + \lambda \int |v_\varepsilon|^2 \ln|v_\varepsilon|^2 - \lambda d\, \|\gamma^2\|_{L^1} \ln\tau$$

$$= \frac{\varepsilon^2}{2} \|\nabla v_{\varepsilon,in}\|_{L^2}^2 + \lambda \int_{\mathbb{R}^d} |v_{\varepsilon,in}|^2 \ln |v_{\varepsilon,in}|^2,$$

Therefore, we obtain thanks to the previous estimate (1.1.6) (along with a Cauchy-Schwarz inequality)

$$\begin{aligned}\left| \frac{\dot\tau^2}{2} \int |y|^2 \, |v_\varepsilon|^2 - \lambda d \, \|\gamma^2\|_{L^1} \ln \tau \right| &\leq \left| \varepsilon \frac{\dot\tau}{\tau} \operatorname{Im} \int v_\varepsilon(t,y) \, y \, \overline{\nabla v_\varepsilon}(t,y) \, \mathrm{d}y \right| \\ &\quad + \left| \frac{\varepsilon^2}{2} \|\nabla v_{\varepsilon,in}\|_{L^2}^2 + \lambda \int_{\mathbb{R}^d} |v_{\varepsilon,in}|^2 \ln |v_{\varepsilon,in}|^2 - \lambda \int |v_\varepsilon|^2 \ln |v_\varepsilon|^2 \right| \\ &\leq \dot\tau(t) \, \|y \, v_\varepsilon(t)\|_{L^2} \frac{\varepsilon}{\tau(t)} \|\nabla v_\varepsilon(t)\|_{L^2} + C_0 \left[\tilde E_\varepsilon^0(v_{\varepsilon,in}) + C\left(\tilde E_\varepsilon^0(v_{\varepsilon,in}) \right) \right] \\ &\leq C\Big(\tilde E_\varepsilon^0(v_{\varepsilon,in}) \Big) \, (\dot\tau(t) + 1).\end{aligned}$$

Multiplying (1.1.3) by $\dot\tau$ and integrating yields

$$\frac{\dot\tau^2}{2} = 2\lambda \ln \tau,$$

which gives in the above inequality for all $t > 1$

$$\left| \int |y|^2 \, |v_\varepsilon|^2 - \frac{d}{2} \|\gamma^2\|_{L^1} \right| \leq C\Big(\tilde E_\varepsilon^0(v_{\varepsilon,in}) \Big) \, \frac{\dot\tau(t) + 1}{\dot\tau^2(t)},$$

and then we can conclude thanks to the identity $\frac{d}{2}\|\gamma^2\|_{L^1} = \int |y|^2 \, \gamma^2(y) \, \mathrm{d}y$. □

Equations on quadratic observables. Finally, we get two equations involving the density and the density of angular momentum defined by

$$\rho_\varepsilon := |v_\varepsilon|^2, \qquad J_\varepsilon := \operatorname{Im}(\varepsilon \, \overline{v_\varepsilon} \, \nabla v_\varepsilon).$$

They satisfy in $\mathcal{D}'((0,\infty) \times \mathbb{R}^d)$

$$\partial_t \rho_\varepsilon + \frac{1}{\tau^2(t)} \nabla \cdot J_\varepsilon = 0, \qquad \partial_t J_\varepsilon + \lambda \, \nabla \rho_\varepsilon + 2\lambda \, y \, \rho_\varepsilon = \frac{\varepsilon^2}{4\,\tau^2(t)} \Delta \nabla \rho_\varepsilon - \frac{\varepsilon^2}{\tau^2(t)} \nabla \cdot (\operatorname{Re}(\nabla v_\varepsilon \otimes \overline{\nabla v_\varepsilon})). \tag{1.3.3}$$

Remark 1.3.4. In the same way as in [33], we can already conclude the weak convergence (in L^1) to γ^2 of $\rho_\varepsilon = |v_\varepsilon|^2$.

Remark 1.3.5. The three most important equations are given by (1.3.2) and (1.3.3). Even though we derive them from the equation (1.3.1) on v_ε in the same way as in [33], it could have been directly derived from some equations for u_ε: the conservation of the mass and the energy and some identities for $|u_\varepsilon|^2$ and $\operatorname{Im}(\varepsilon \, \overline{u_\varepsilon} \, \nabla u_\varepsilon)$ similar to (1.3.3) (and some other estimates which arise from them, like the conservation of the angular momentum, the variation of the second momentum and the variation of $\int_{\mathbb{R}^d} x \operatorname{Im}(\varepsilon \, \overline{u_\varepsilon} \, \nabla u_\varepsilon) \, \mathrm{d}x$):

$$\begin{aligned}\partial_t (|u_\varepsilon|)^2 + \nabla \cdot \operatorname{Im}(\varepsilon \, \overline{u_\varepsilon} \, \nabla u_\varepsilon) = 0, \\ \partial_t \left(\operatorname{Im}(\varepsilon \, \overline{u_\varepsilon} \, \nabla u_\varepsilon) \right) + \lambda \, \nabla(|u_\varepsilon|^2) = \frac{\varepsilon^2}{4} \Delta \nabla(|u_\varepsilon|^2) - \varepsilon^2 \, \nabla \cdot (\operatorname{Re}(\nabla u_\varepsilon \otimes \overline{\nabla u_\varepsilon})).\end{aligned} \tag{1.3.4}$$

This is an important remark in view of Section 1.5.

1.3.2. Proof of Corollary 1.1.9

Again, this proof is extremely similar to that in [33]. In the energy for u_ε, write the potential energy in terms of v_ε.

$$\begin{aligned}\int_{\mathbb{R}^d} |u_\varepsilon(t)|^2 \ln |u_\varepsilon(t)|^2 &= -d \left(\ln \tau(t) + \ln \left(\frac{\|u_{\varepsilon,\text{in}}\|_{L^2}^2}{\|\gamma^2\|_{L^1}} \right) \right) \|u_{\varepsilon,\text{in}}\|_{L^2}^2 + \frac{\|u_{\varepsilon,\text{in}}\|_{L^2}^2}{\|\gamma^2\|_{L^1}} \int |v_\varepsilon(t)|^2 \ln |v_\varepsilon(t)|^2 \\ &= -d \|u_{\varepsilon,\text{in}}\|_{L^2}^2 \ln \tau(t) + \mathrm{O}(1).\end{aligned}$$

The conservation of the energy for u_ε yields

$$\varepsilon^2 \, \|\nabla u_\varepsilon\|_{L^2}^2 \underset{t \to \infty}{\sim} 2\lambda d \|u_{\varepsilon,\text{in}}\|_{L^2}^2 \ln \tau(t).$$

Now fix $0 < \delta < 1$. By interpolation, we readily have

$$\varepsilon^\delta \|u_\varepsilon(t)\|_{\dot H^\delta} \lesssim \|u_\varepsilon\|_{L^2}^{1-\delta} \, (\varepsilon^2 \, \|u_\varepsilon(t)\|_{\dot H^1}^\delta) \lesssim (\ln t)^{\frac{\delta}{2}}.$$

For the other inequality, we recall the lemma used in [33] without semiclassical constant. However, in our context, it is better to recall it with semiclassical constant.

Lemma 1.3.6 ([1, Lemma 1.5.]). *There exists a constant $C > 0$ such that for all $\varepsilon \in (0,1]$, for all $\delta \in [0,1]$, for all $u \in H^1(\mathbb{R}^d)$ and for all $w \in \dot{W}^{1,\infty}(\mathbb{R}^d)$,*

$$\| \, |w|^\delta u\|_{L^2} \leq \varepsilon^\delta \|u\|_{\dot{H}^\delta} + \|u\|_{L^2}^{1-\delta} \, \|(\varepsilon\nabla - iw)u\|_{L^2}^\delta + C\, \varepsilon^{\frac{\delta}{2}} \, (1 + \|\nabla w\|_{L^\infty}) \, \|u\|_{L^2}.$$

Applying this lemma with $u_\varepsilon(t)$ and

$$w(t,x) := \frac{\dot{\tau}(t)}{\tau(t)} \, x,$$

we get for all $t \geq 0$

$$\dot{\tau}(t)^\delta \, \| \, |y|^\delta \, v_\varepsilon(t)\|_{L^2} \leq \|u_\varepsilon\|_{\dot{H}^\delta} + \left\| \frac{\varepsilon}{\tau(t)} \nabla v_\varepsilon(t) \right\|_{L^2}^\delta \frac{\|u_{\varepsilon,\text{in}}\|_{L^2}}{\|\gamma\|_{L^2}^\delta} + C\, \varepsilon^{\frac{\delta}{2}} \left(1 + \frac{\dot{\tau}(t)}{\tau(t)} \right) \|u_{\varepsilon,\text{in}}\|_{L^2}.$$

The result readily follows: all the terms of the right hand side are bounded but the first one, and the behavior of the left hand side is given by the convergence in Wasserstein distance W_2 which implies (since $\delta \in (0,1)$)

$$\int_{\mathbb{R}^d} |y|^{2\delta} \, |v_\varepsilon(t,y)|^2 \, \mathrm{d}y \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} |y|^{2\delta} \, \gamma^2(y) \, \mathrm{d}y.$$

1.3.3. First part of the proof of Theorem 1.1.10

From now on, C_0 denotes a positive constant (which may change from line to line) independent of t and ε and we assume that (A1) is satisfied.

Convergence of the Wigner Transforms and first properties. First, we proved that $(v_\varepsilon)_{\varepsilon>0}$ satisfies (A3) thanks to (1.1.6), hence we can apply Lemma 1.2.3 for $(v_\varepsilon)_{\varepsilon>0}$ and also for $(u_\varepsilon)_{\varepsilon>0}$ for all $T > 0$ because $(u_\varepsilon)_{\varepsilon>0}$ satisfies (A3) thanks to (1.1.5) and the first assumption of (A1). By an argument of diagonal extraction, it leads to a subsequence (still denoted ε) and two measures $W = W(t,x,\xi)$ and $\tilde{W} = \tilde{W}(t,y,\eta)$ in $L^\infty((0,\infty), \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d))$ such that for every $p \in [1,\infty)$

$$\begin{gathered} W_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} W, \qquad \tilde{W}_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \tilde{W}, \qquad \text{in } L^p_{\text{loc}}((0,\infty), \mathcal{A}'), \\ \rho_\varepsilon = |v_\varepsilon|^2 \underset{\varepsilon\to 0}{\longrightarrow} \tilde{\rho} := \int_{\mathbb{R}^d} \tilde{W}(.,.,\mathrm{d}\eta) \qquad \text{in } L^p_{\text{loc}}\left((0,\infty), \mathcal{M}(\mathbb{R}^d)\right), \\ \iint_{\mathbb{R}^d\times\mathbb{R}^d} |y|^2 \, \tilde{W}(t,\mathrm{d}y,\mathrm{d}\eta) \leq C_0, \qquad \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\eta|^2 \, \tilde{W}(t,\mathrm{d}y,\mathrm{d}\eta) \leq C_0 \, \tau(t)^2. \end{gathered} \tag{1.3.5}$$

But (1.1.6) also gives that ρ_ε is uniformly bounded in $L^\infty\left((0,\infty), L\log L(\mathbb{R}^d)\right)$. Therefore, $\tilde{\rho} \in L^\infty\left((0,\infty), L\log L(\mathbb{R}^d)\right)$. It remains to prove that $\tilde{\rho} \in \mathcal{C}([0,\infty), \mathcal{P}_1(\mathbb{R}^d))$. Come back to the equation for $\partial_t \rho_\varepsilon$ in (1.3.3):

$$\partial_t \rho_\varepsilon + \frac{1}{\tau^2(t)} \nabla \cdot J_\varepsilon = 0 \qquad \text{in } \mathcal{D}',$$

where we recall $J_\varepsilon = \operatorname{Im}(\varepsilon \, \overline{v_\varepsilon} \nabla v_\varepsilon)$. We also recall that $\frac{1}{\tau(t)} J_\varepsilon$ is bounded in $L^\infty((0,\infty), L^1_1(\mathbb{R}^d))$ uniformly in $\varepsilon > 0$ thanks to (1.1.6) and a Cauchy-Schwarz inequality. Therefore, thanks to Kantorovich-Rubinstein duality, $\pi^{-\frac{d}{2}} \, \rho_\varepsilon$ is equicontinuous with values in $\mathcal{P}_1(\mathbb{R}^d)$ (endowed with the Wasserstein metric $\mathcal{W}_1$). Moreover, since ρ_ε is bounded in $L^\infty\left((0,\infty), L^1_2 \cap L\log L(\mathbb{R}^d)\right)$, de la Vallée-Poussin and Dunford-Pettis theorems yield the compactness of $\left\{\pi^{-\frac{d}{2}} \, \rho_\varepsilon(t), \varepsilon > 0\right\}$ in $\mathcal{P}_1(\mathbb{R}^d)$ for all $t \geq 0$. Hence, by Ascoli theorem, $\left\{\pi^{-\frac{d}{2}} \, \rho_\varepsilon\right\}$ is a compact set in $\mathcal{C}\left([0,T], \mathcal{P}_1(\mathbb{R}^d)\right)$ for all $T > 0$. Thus we get not only $\pi^{-\frac{d}{2}} \, \tilde{\rho} \in \mathcal{C}([0,\infty), \mathcal{P}_1(\mathbb{R}^d))$ but also

$$\pi^{-\frac{d}{2}} \, \rho_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \pi^{-\frac{d}{2}} \, \tilde{\rho} \qquad \text{in } \mathcal{C}\left([0,T], \mathcal{P}_1(\mathbb{R}^d)\right) \text{ for all } T > 0.$$

Moreover, the identity $\tilde{\rho}(t,\mathbb{R}^d) = \pi^{\frac{d}{2}}$, satisfied for a.e. $t \in (0,\infty)$ thanks to Lemma 1.2.3, is actually satisfied for *all* $t \in [0,\infty)$.

Lastly, an easy computation leads to the relation (1.1.12), substituting u_ε by v_ε thanks to (1.1.5) and performing a simple change of variable. In this relation (1.1.12), two terms already converges: $\tilde{W}_\varepsilon$ converges to the non-null measure $\tilde{W}$ and W_ε converges to the measure W. Therefore, thanks to this relation, it is easy to prove that $\|u_{\varepsilon,\text{in}}\|_{L^2}^2$ converges. Thus we can pass to the limit in the relation between the two Wigner Transforms.

Estimates on the momenta. We already proved the estimates (1.3.5). Moreover, in the same way, (1.1.7) can be translated into a property on the Husimi Transform thanks to (1.2.3):

$$\int_0^\infty \frac{\dot\tau(t)}{\tau^3(t)} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |\eta|^2\, \tilde W_\varepsilon^H(t,\mathrm{d}y,\mathrm{d}\eta)\,\mathrm{d}t \le C_0.$$

A slight modification of the proof of Lemma 1.2.3 shows that this estimate still holds after passing to the limit, so that we get (1.1.13). For (1.1.14), we proved that ρ_ε converges locally uniformly in time to $\tilde\rho$ and has a second momentum bounded uniformly in $t > 0$ and $\varepsilon > 0$. Therefore, a usual argument shows that

$$\int_{\mathbb{R}^d} y\,\rho_\varepsilon(t,y)\,\mathrm{d}y \underset{\varepsilon\to0}{\longrightarrow} \int_{\mathbb{R}^d} y\,\tilde\rho(t,y)\,\mathrm{d}y \qquad \text{locally uniformly in } t.$$

However, the term on the left-hand side has already been computed in (1.1.8), hence the affine function $\frac{\|\gamma^2\|_{L^1}}{\|u_{\varepsilon,\mathrm{in}}\|_{L^2}^2}\,(I_{1,0}^\varepsilon\, t + I_{2,0}^\varepsilon)$ converges locally uniformly in t. Thus, we conclude that both $\frac{\|\gamma^2\|_{L^1}}{\|u_{\varepsilon,\mathrm{in}}\|_{L^2}^2}\, I_{1,0}^\varepsilon$ and $\frac{\|\gamma^2\|_{L^1}}{\|u_{\varepsilon,\mathrm{in}}\|_{L^2}^2}\, I_{2,0}^\varepsilon$ converge to some C_1 and C_2, and then we obtain (1.1.14). This completes the first part of the proof.

1.3.4. Convergence of some other quantities

Actually, we would like to pass to the limit in the two identities in (1.3.3). For this, there are still two quantities which should converge: J_ε and $\varepsilon^2\ \mathrm{Re}\left(\nabla v_\varepsilon \otimes \overline{\nabla v_\varepsilon}\right)$. First, we recall the estimates found for those two quantities thanks to (1.1.6) and (1.1.7) (up to a Cauchy-Schwarz inequality for J_ε)

$$\frac{1}{\tau(t)}\int_{\mathbb{R}^d}(1+|y|)\,|J_\varepsilon(t,y)|\,\mathrm{d}y \le C_0 \quad \text{for all } t\ge 0 \text{ and } \varepsilon>0,$$

$$\int_0^\infty \frac{\dot\tau(t)}{\tau(t)}\left(\frac{1}{\tau(t)}\int_{\mathbb{R}^d}(1+|y|)\,|J_\varepsilon(t,y)|\,\mathrm{d}y\right)^2 \mathrm{d}t \le C_0, \tag{1.3.6}$$

$$\frac{\varepsilon^2}{\tau^2(t)}\left\|\mathrm{Re}(\nabla v_\varepsilon(t)\otimes\overline{\nabla v_\varepsilon(t)})\right\|_{L^1} \le C_0 \qquad \text{for all } t\ge 0 \text{ and } \varepsilon>0,$$

$$\int_0^\infty \frac{\dot\tau(t)}{\tau(t)}\int_{\mathbb{R}^d}\left|\frac{\varepsilon^2}{\tau^2(t)}\,\mathrm{Re}(\nabla v_\varepsilon(t)\otimes\overline{\nabla v_\varepsilon(t)})\right|\mathrm{d}y\,\mathrm{d}t \le C_0. \tag{1.3.7}$$

Moreover, J_ε and $\varepsilon^2\ \mathrm{Re}\left(\nabla v_\varepsilon \otimes \overline{\nabla v_\varepsilon}\right)$ are related to the Husimi Transform respectively through (1.2.7) and (1.2.9). An analysis similar to that for the density and for the second momentum for the Wigner Measure in the proof of Lemma 1.2.3 shows that for all $p > 1$:

$$\frac{1}{\tau(t)}\,J_\varepsilon \underset{\varepsilon\to0}{\longrightarrow} \mu_0 := \frac{1}{\tau(t)}\int_{\mathbb{R}^d}\eta\,\tilde W(t,y,\mathrm{d}\eta) \quad \text{in } L^p_{\mathrm{loc}}((0,\infty),\mathcal{M}^s(\mathbb{R}^d)^d),$$

$$\frac{\varepsilon^2}{\tau^2(t)}\ \mathrm{Re}\left(\nabla v_\varepsilon\otimes\overline{\nabla v_\varepsilon}\right) \underset{\varepsilon\to0}{\rightharpoonup} \nu_0 := \frac{1}{\tau^2(t)}\int_{\mathbb{R}^d}\eta\otimes\eta\,\tilde W(t,y,\mathrm{d}\eta) \quad \text{in } L^p_{\mathrm{loc}}\left((0,\infty),\mathcal{M}^s\left(\mathbb{R}^d\right)^{d\times d}\right),$$

where $\mathcal{M}^s(\mathbb{R}^d)$ is the set of finite signed measure on $\mathbb{R}^d$. In particular,

$$\sup_{t>0}\operatorname{ess}\int_{\mathbb{R}^d}(1+|y|)\,|\mu_0|(t,\mathrm{d}y) + \int_0^\infty\frac{\dot\tau(t)}{\tau(t)}\left(\int_{\mathbb{R}^d}(1+|y|)\,|\mu_0|(t,\mathrm{d}y)\right)^2\mathrm{d}t \le C_0, \tag{1.3.8}$$

$$\sup_{t>0}\operatorname{ess}|\nu_0|(t,\mathbb{R}^d) + \int_0^\infty\frac{\dot\tau(t)}{\tau(t)}\,|\nu_0|(t,\mathbb{R}^d)\,\mathrm{d}t \le C_0, \tag{1.3.9}$$

where $|\mu_0|$ (resp. $|\nu_0|$) is the absolute variation of μ_0 (resp ν_0), and (1.3.3) becomes

$$\partial_t\tilde\rho + \frac{1}{\tau(t)}\nabla\cdot\mu_0 = 0, \qquad \partial_t(\tau(t)\,\mu_0) + \lambda\,\nabla\tilde\rho + 2\lambda\,y\,\tilde\rho = -\nabla\cdot\nu_0, \qquad \text{in } \mathcal{D}'((0,\infty)\times\mathbb{R}^d). \tag{1.3.10}$$

Remark 1.3.7. The latter equation along with the estimates (1.1.13), (1.3.8) and (1.3.9) gives some new estimate for $\tau(t)\,\mu_0$, due to the fact that $\partial_t(\tau(t)\,\mu_0) \in L^\infty\left((0,\infty), W^{-1-\delta,1}(\mathbb{R}^d)\right)$ for all $\delta > 0$, uniformly in δ:

$$\|\partial_t(\tau(t)\,\mu_0)\|_{L^\infty_t W^{-1-\delta,1}_y} \le \lambda\|\tilde\rho\|_{L^\infty_t L^1_y} + 2\lambda\|y\,\tilde\rho\|_{L^\infty_t L^1_y} + \sup_{t>0}\operatorname{ess}|\nu_0|(t,\mathbb{R}^d) \le C_0.$$

In particular, $J_0 := \tau(t)\,\mu_0 \in \mathcal{C}\left([0,\infty), W^{-1-\delta,1} \cap \mathcal{M}^s(\mathbb{R}^d)\right)$ for all $\delta > 0$, so that μ_0 is actually defined for *all* $t \geq 0$. We can also derive that

$$\|J_0(t_0 + t, y) - J_0(t, y)\|_{W_y^{-1-\delta,1}} \leq C_0\, t_0, \qquad \forall t \geq 0,\, \forall t_0 \geq 0.$$

But $J_0(t) \in \mathcal{M}^s(\mathbb{R}^d) \subset W^{-1,1}(\mathbb{R}^d)$ for all $t \geq 0$. Therefore, for all $t, t_0 \geq 0$, we can take the limit $\delta \to 0$ and we get

$$\|J_0(t_0 + t, y) - J_0(t, y)\|_{W_y^{-1,1}} \leq C_0\, t_0, \qquad \forall t \geq 0,\, \forall t_0 \geq 0.$$

In particular, this leads to:

$$\|\mu_0(t)\|_{W^{-1,1}} \leq C_0\, \frac{1+t}{\tau(t)} \underset{t\to\infty}{\longrightarrow} 0, \qquad \forall t \geq 0.$$

1.4. Fokker-Planck equation and convergence rate in Wasserstein distance

1.4.1. From Schrödinger to Fokker-Planck

We define $\rho_0 := \tilde{\rho}$, $\mu_\varepsilon := \frac{1}{\tau(t)}\, J_\varepsilon$ and $\nu_\varepsilon := \frac{\varepsilon^2}{\tau^2(t)}\, \operatorname{Re}\left(\nabla v_\varepsilon \otimes \overline{\nabla v_\varepsilon}\right)$ for all $\varepsilon > 0$, so that we can write (1.3.3) and (1.3.10) in a single generalized system for all $\varepsilon \geq 0$ (which also holds in $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$):

$$\partial_t \rho_\varepsilon + \frac{1}{\tau(t)} \nabla \cdot \mu_\varepsilon = 0, \qquad \partial_t(\tau(t)\mu_\varepsilon) + \lambda\,\nabla\rho_\varepsilon + 2\lambda\, y\,\rho_\varepsilon = \frac{\varepsilon^2}{4\,\tau^2(t)} \Delta\nabla\rho_\varepsilon - \nabla\cdot\nu_\varepsilon. \tag{1.4.1}$$

In a similar way as in [33, Theorem 1.7.], combining those two equations leads to

$$\partial_t(\tau^2\,\partial_t\rho_\varepsilon) = \lambda\, L\rho_\varepsilon - \frac{\varepsilon^2}{4\,\tau^2(t)}\Delta^2\rho_\varepsilon + \nabla\cdot(\nabla\cdot\nu_\varepsilon)\,,$$

where $L = \Delta + \nabla\cdot(2y\,.)$ is a Fokker-Planck operator. On the other hand,

$$\partial_t(\tau^2\,\partial_t\rho_\varepsilon) = \tau^2\,\partial_t^2\rho_\varepsilon + 2\dot{\tau}\tau\,\partial_t\rho_\varepsilon.$$

Since $\tau^2 \ll (\dot{\tau}\tau)^2$, it is natural to change scales in time and define

$$s = \int \frac{\lambda}{\dot{\tau}\tau} = \int \frac{\ddot{\tau}}{2\,\dot{\tau}} = \frac{1}{2}\ln\dot{\tau}(t). \tag{1.4.2}$$

From this we have (using the notation $f(t) = \check{f}(s(t))$ for the change of time variable)

$$\partial_s\check{\rho}_\varepsilon - \frac{2\lambda}{\dot{\check{\tau}}^2}\partial_s\check{\rho}_\varepsilon + \frac{\lambda}{\dot{\check{\tau}}^2}\partial_s^2\check{\rho}_\varepsilon = L\check{\rho}_\varepsilon - \frac{\varepsilon^2}{4\,\lambda\,\check{\tau}^2(s)}\Delta^2\check{\rho}_\varepsilon + \frac{1}{\lambda}\nabla\cdot(\nabla\cdot\check{\nu}_\varepsilon)\,, \tag{1.4.3}$$

Discarding formally negligible terms leads to the Fokker-Planck equation without source terms

$$\partial_s\check{\rho}_\varepsilon = L\check{\rho}_\varepsilon,$$

for which it is well-known (see for instance [9, Corollary 2.17.]) that in large times the solution converges strongly in L^1 to an element of the kernel of L, hence a Gaussian. Notice that the convergence is exponentially fast in s variable, so coming back into t variable produces a logarithmic decay (which is exactly what we are expecting) due to the estimate

$$s = \frac{1}{4}\ln\ln t + \mathrm{o}(1) \qquad \text{as } t \to \infty.$$

In particular, translating the properties of convergence (1.1.10) and (1.1.15) in terms of s leads to

$$\mathcal{W}_1\left(\pi^{-\frac{d}{2}}\,\check{\rho}_\varepsilon(s), \pi^{-\frac{d}{2}}\,\gamma^2\right) \leq C_0\, e^{-2s}, \qquad \forall s > 0. \tag{1.4.4}$$

It is worth mentioning that exponential convergence also occurs in 2-Wasserstein distance for Fokker-Planck equations without source terms (see for instance [24, 145]). In particular, for our particular Fokker-Planck operator, such a result reads as follows:

Lemma 1.4.1 ([24, 145])**.** *For any $f_{\text{in}} \in \mathcal{P}_2 \cap L\log L\,(\mathbb{R}^d)$, the solution f to $\partial_t f = Lf$ with $f(0) = f_{\text{in}}$ satisfies:*

$$\mathcal{W}_2\left(f(t), \pi^{-\frac{d}{2}}\,\gamma^2\right) \leq e^{-2t}\,\mathcal{W}_2\left(f_{\text{in}}, \pi^{-\frac{d}{2}}\,\gamma^2\right), \qquad \forall t \geq 0.$$

Therefore, the s variable must be better suited for our study. The following lemma computes $\check{\dot{\tau}}$ and $\check{\tau}$, which will be needed in the rest of the paper.

Lemma 1.4.2. *With the previous notations, for all $s \in \mathbb{R}$:*

$$\check{\tau}(s) = \exp\left[\frac{e^{4s}}{4\lambda}\right], \qquad \check{\dot{\tau}}(s) = e^{2s}.$$

Proof. The second identity is easy to state thanks to (1.4.2). Then the same change of variable allows us to compute:

$$\frac{\mathrm{d}\check{\tau}}{\mathrm{d}s}(s) = \left(\frac{\tau\,\dot{\tau}}{\lambda}\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)(t(s)) = \left(\frac{\tau\,\dot{\tau}^2}{\lambda}\right)(t(s)) = \frac{e^{4s}}{\lambda}\check{\tau}(s).$$

This yields the first identity thanks to the fact that $\lim_{s\to-\infty}\check{\tau}(s) = \lim_{t\to 0^+}\tau(t) = \tau(0) = 1$. □

Actually, we prove a slightly better result which may be adapted for other situations (for instance for Section 1.5).

Lemma 1.4.3. *Let $g_1 \in L^\infty\left((0,\infty),\mathcal{M}^s(\mathbb{R}^d)^d\right)$, $g_2 \in \mathcal{C}_b\left([0,\infty),\mathcal{M}^s(\mathbb{R}^d)^d\right) \cap L^\infty\left((0,\infty),\mathcal{M}^s_1(\mathbb{R}^d)^d\right)$, $g_3 \in L^\infty\left((0,\infty),\mathcal{M}^s(\mathbb{R}^d)\right)$ and $g_4 \in L^\infty\left((0,\infty),\mathcal{M}^s(\mathbb{R}^d)^{d\times d}\right)$ such that*

$$G := \|g_2\|_{L^\infty_t\mathcal{M}} + \|e^{6t}\,g_3\|_{L^\infty_t\mathcal{M}} + \|g_4\|_{L^\infty_t\mathcal{M}} + \sup_{t>0}\text{ess}\int_{\mathbb{R}^d}|y|\,|g_2|(t,\mathrm{d}y) + \left(\int_0^\infty\left(e^{2t}\,|g_1|(t,\mathbb{R}^d)\right)^2\mathrm{d}t\right)^{\frac12} + \left(\int_0^\infty\left(e^{2t}\,|g_2|(t,\mathbb{R}^d)\right)^2\mathrm{d}t\right)^{\frac12} + \int_0^\infty e^{4t}\,|g_4|\,(t,\mathbb{R}^d)\,\mathrm{d}t < \infty. \quad (1.4.5)$$

Let $f_{\text{in}} \in \mathcal{P}_2 \cap L\log L(\mathbb{R}^d)$ and suppose there exists $f := f(t,y) \in L^\infty((0,\infty),\mathcal{P}_2(\mathbb{R}^d)) \cap \mathcal{C}([0,\infty),L^1_w(\mathbb{R}^d))$ satisfying

$$\partial_t f = Lf + e^{-2t}\,\nabla\cdot g_1 + \partial_t(e^{-2t}\,\nabla\cdot g_2) + \Delta^2 g_3 + \nabla\cdot(\nabla\cdot g_4), \qquad f(0) = f_{\text{in}}. \quad (1.4.6)$$

Then there exists $C > 0$ such that

$$\mathcal{W}_1(f(t),\pi^{-\frac{d}{2}}\gamma^2) \le C\,(1 + G + \||y|^2 f\|_{L^\infty_t L^1_y})\,e^{-2t} \qquad \forall t > 0.$$

This result shows that if we already have some estimates for the function solution to the Fokker-Planck equation with source terms, and if the source terms are negligible enough, then the main behavior coming from the Fokker-Planck equation *without* source terms still holds for this function. It is actually related to the very particular form of the Fokker-Planck operator we are considering. In the same way as above with the transformation from (1.4.1) to (1.4.3), such a result may be expressed with a system similar to (1.4.1).

Lemma 1.4.4. *Let $\lambda > 0$, $h_1 \in \mathcal{C}_b\left((0,\infty),\mathcal{M}^s(\mathbb{R}^d)^d\right) \cap L^\infty\left((0,\infty),\mathcal{M}^s_1(\mathbb{R}^d)^d\right)$, $h_2 \in L^\infty\left((0,\infty),\mathcal{M}^s(\mathbb{R}^d)\right)$ and $h_3 \in L^\infty\left((0,\infty),\mathcal{M}^s(\mathbb{R}^d)^{d\times d}\right)$ such that*

$$G_0 := \|h_1\|_{L^\infty\mathcal{M}} + \|h_2\|_{L^\infty\mathcal{M}} + \|h_3\|_{L^\infty\mathcal{M}} + \sup_{t>0}\text{ess}\int_{\mathbb{R}^d}|y|\,|h_1|(t,\mathrm{d}y) + \left(\int_0^\infty\frac{\dot{\tau}(t)}{\tau(t)}\,\left(|h_1|(t,\mathbb{R}^d)\right)^2\mathrm{d}t\right)^{\frac12} + \int_0^\infty\frac{\dot{\tau}(t)}{\tau(t)}\,|h_3|\,(t,\mathbb{R}^d)\,\mathrm{d}t < \infty. \quad (1.4.7)$$

Suppose there exists $f := f(t,y) \in L^\infty((0,\infty),\mathcal{P}_2 \cap L\log L(\mathbb{R}^d)) \cap \mathcal{C}((0,\infty),L^1_w(\mathbb{R}^d))$ satisfying

$$\partial_t f + \frac{1}{\tau(t)}\nabla\cdot h_1 = 0, \qquad \partial_t(\tau(t)h_1) + \lambda\,\nabla f + 2\lambda\,y\,f = \frac{1}{\tau(t)^2}\Delta\nabla h_2 - \nabla\cdot h_3. \quad (1.4.8)$$

Then there exists $C > 0$ depending only on λ such that

$$\mathcal{W}_1(f(t),\pi^{-\frac{d}{2}}\gamma^2) \le C\,\frac{1 + G_0 + \||y|^2 f\|_{L^\infty_t L^1_y}}{\dot{\tau}(t)}, \qquad \forall t > 1.$$

Remark 1.4.5. The assumption $f \in \mathcal{C}((0,\infty),L^1_w(\mathbb{R}^d))$ can be removed since it easily follows from the first equation in (1.4.8) and the fact that $h_1 \in L^\infty\left((0,\infty),\mathcal{M}^s(\mathbb{R}^d)^d\right)$.

Proof. In the same way as above, combining both equations in (1.4.8) with the change of time variable $s = \frac{1}{2}\ln\dot{\tau}(t)$ (with the same notation for this change of variable in the functions) leads to the equation

$$\partial_s \check{f} - \frac{2\lambda}{\check{\dot{\tau}}^2}\partial_s \check{f} + \frac{\lambda}{\check{\dot{\tau}}^2}\partial_s^2 \check{f} = L\check{f} - \frac{1}{\lambda\,\check{\tau}^2(s)}\Delta^2 \check{h}_2 + \frac{1}{\lambda}\nabla\cdot\check{h}_3. \tag{1.4.9}$$

The first equation of (1.4.8) reads in terms of s

$$\partial_s \check{f} + \frac{\check{\dot{\tau}}(s)}{\lambda}\,\nabla\cdot\check{h}_1 = 0.$$

Substituting $\partial_s \check{f}$ in the second and third term of the left-hand side of (1.4.9), we compute

$$\frac{1}{\check{\dot{\tau}}^2}\partial_s(\check{\dot{\tau}}\,\check{h}_1) = e^{-4s}\partial_s(e^{2s}\,\check{h}_1) = \partial_s(e^{-2s}\,\check{h}_1) + 4e^{-2s}\,\check{h}_1,$$

and so

$$\partial_s \check{f} = L\check{f} + 2\,e^{-2s}\,\nabla\cdot\check{h}_1 + \partial_s(e^{-2s}\,\nabla\cdot\check{h}_1) - \frac{1}{\lambda\,\check{\tau}(s)^2}\Delta^2\check{h}_2 + \frac{1}{\lambda}\nabla\cdot\left(\nabla\cdot\check{h}_3\right).$$

Hence we can apply Lemma 1.4.3 with $g_1 = 2\,g_2 = \check{h}_1$, $g_3 = -\frac{1}{\lambda\,\check{\tau}(s)^2}\,\check{h}_2$ and $g_4 = \frac{1}{\lambda}\check{h}_3$ since the translation of (1.4.7) into the s variable implies $G \le C\,G_0$ for some C depending only on λ. The inequality we get from its conclusion leads to the expected result when coming back into the t variable. □

The results (1.1.10) and (1.1.15) follow then as a simple corollary.

Corollary 1.4.6. *Given any $\lambda > 0$ and $(u_{\varepsilon,\mathrm{in}})_{\varepsilon>0}$ satisfying* (A1)*, define u_ε and v_ε provided by Theorems 1.1.4 and 1.1.5 and set $\rho_\varepsilon := |v_\varepsilon|^2$ for all $\varepsilon > 0$. For $\varepsilon = 0$, define also ρ as the density of a Wigner Measure of the sequence $(u_\varepsilon)_{\varepsilon>0}$ given by Theorem 1.1.10 and set $\rho_0 = \rho$.*

Then there exists $C > 0$ depending only on d and λ such that for all $\varepsilon \in [0,1]$,

$$\mathcal{W}_1\left(\pi^{-\frac{d}{2}}\rho_\varepsilon(t), \pi^{-\frac{d}{2}}\gamma^2\right) \le \frac{C}{\sqrt{\ln t}}, \qquad \forall t \ge 2.$$

Proof. The estimates (1.3.6)-(1.3.9) read in the s variable:

$$\int_0^\infty \left(e^{2s}\,|\mu_\varepsilon|(s,\mathbb{R}^d)\right)^2 \mathrm{d}s + \int_0^\infty e^{4s}\,|\nu_\varepsilon|\,(s,\mathbb{R}^d)\,\mathrm{d}s \le C_0, \qquad \forall\varepsilon \ge 0.$$

Since (1.4.1) holds we can apply Lemma 1.4.4 with (up to a factor $\pi^{-\frac{d}{2}}$) $f = \rho_\varepsilon$, $h_1 = \mu_\varepsilon$, $h_2 = \frac{\varepsilon^2}{4}\,\check{\rho}_\varepsilon$ and $h_3 = \nu_\varepsilon$ where G_0 (defined in (1.4.7)) is uniformly bounded in ε thanks to the previous result along with the estimates already proven in Theorems 1.1.5 and 1.1.10. We also know that the second momentum (in space) of ρ_ε is bounded uniformly in time and in ε. The result leads to (1.4.4) which establishes the formula when coming back to the t variable. □

1.4.2. The harmonic Fokker-Planck operator

The Fokker-Planck operator $L = \Delta + \nabla\cdot(2y\,.)$ is very special and well-known, due in particular to its links with the heat equation. Its form allows to compute explicitly its kernel and therefore get better estimates for the solution. Those estimates will be helpful in order to compute some convergence rate.

The fact that the kernel can be computed explicitly comes from the fact that taking the Fourier Transform in space of the Fokker-Planck operator transforms it into a simple transport operator with a source term $-|\xi|^2\,\hat{f}$ which leads to a simple first order ODE when applying the method of characteristics, with the notation $\hat{f}$ for the spatial Fourier Transform of any function $f = f(s,y)$. This operator is also related to the heat equation. Indeed, if $H = H(t,x)$ is a solution to the heat equation $\partial_t H = \frac{1}{2}\Delta H$, then $f = f(t,x)$ defined by

$$f(t,x) = e^{2dt}\,H\left(\frac{e^{4t}-1}{2}, e^{2t}\,x\right), \qquad \forall t \ge 0, \forall x \in \mathbb{R}^d, \tag{1.4.10}$$

is a solution to the harmonic Fokker-Planck equation $\partial_t f = Lf$. The inverse change of variable allows to pass from the heat equation to the harmonic Fokker-Planck equation in the same way. Its kernel is therefore easy to compute.

Lemma 1.4.7. *The kernel $\mathcal{K} = \mathcal{K}(t,x,\xi)$ of the harmonic Fokker-Planck semi-group is given by*

$$\mathcal{K}(t,x,y) := \pi^{-\frac{d}{2}}\left(1-e^{-4t}\right)^{-\frac{d}{2}}\,\gamma^2\Big((x - e^{-2t}y)\left(1-e^{-4t}\right)^{-\frac{1}{2}}\Big).$$

Proof. For any $f_{\rm in} \in \mathcal{S}(\mathbb{R}^d)$, we want to compute the solution f to $\partial_t f = Lf$ with initial data $f(0) = f_{\rm in}$. The function H defined by the rescaling (1.4.10) is solution of the heat equation with initial data $H(0) = f_{\rm in}$, therefore it is known that for all $t > 0$ and $x \in \mathbb{R}^d$

$$H(t,x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{2t}\right) f_{\rm in}(y)\,\mathrm{d}y.$$

The result follows from some basic computations. □

The kernel for the harmonic Fokker-Planck equation is of course very similar to that for the heat equation. In particular, for all $t > 0$ and all $x \in \mathbb{R}^d$, $\mathcal{K}(t,x,.) \in \mathcal{S}(\mathbb{R}^d)$, so there is a huge regularization in the same way as for the heat equation. Moreover, if e^{tL} is not a convolution (which is the case for the heat equation), it is not far from this feature since $\mathcal{K}(t)$ depends only on $x - e^{-2t}y$. In particular, we get for all $n \in \mathbb{N}$, $I \in \{1,...,d\}^n$, $t > 0$ and $x, y \in \mathbb{R}^d$,

$$\partial_{y_I}\mathcal{K}(t,x,y) = (-1)^n e^{-2nt}\,\partial_{x_I}\mathcal{K}(t,x,y), \quad \text{and} \quad \int_{\mathbb{R}^d} |D_x^n \mathcal{K}(t,x,y)|\,\mathrm{d}x = \frac{\left(1-e^{-4t}\right)^{-\frac{n}{2}}}{\pi^{\frac{d}{2}}} \int_{\mathbb{R}^d} \left|D^n(\gamma^2)(x)\right|\mathrm{d}x.$$

There is also another identity we will need later:

$$\int_{\mathbb{R}^d} |x - e^{-2t}y|\;|D_x^n\mathcal{K}(t,x,y)|\,\mathrm{d}x = \frac{\left(1-e^{-4t}\right)^{-\frac{n-1}{2}}}{\pi^{\frac{d}{2}}} \int_{\mathbb{R}^d} |x|\left|D^n(\gamma^2)(x)\right|\mathrm{d}x. \tag{1.4.11}$$

The first two identities are crucial for the next lemma.

Lemma 1.4.8. *Given any $f_0 \in \mathcal{M}^s(\mathbb{R}^d)$, $n \in \mathbb{N}$ and $I \in \{1,...,d\}^n$, $f(t) = e^{tL}(\partial_I^n f_0)$ is a $W^{\infty,1}$ function for all $t > 0$, and for all $m \in \mathbb{N}$ we have:*

$$f(t) = e^{-2nt}\,\partial_I^n\left(e^{tL}f_0\right) \qquad \text{and} \qquad \|f(t)\|_{\dot{W}^{m-n,1}} \leq \frac{e^{-2nt}}{\left(1-e^{-4t}\right)^{\frac{m}{2}}}\,\frac{\|\gamma^2\|_{\dot{W}^{m,1}}}{\pi^{\frac{d}{2}}}\,|f_0|(\mathbb{R}^d).$$

Proof. With n integrations by parts, the previous identity for $\partial_{y_I}\mathcal{K}(t,x,.)$ and the Lebesgue theorem, we get for all $t > 0$ and $x \in \mathbb{R}^d$:

$$f(t,x) = \langle \mathcal{K}(t,x,.), \partial_I f_0\rangle = (-1)^n\langle \partial_{y_I}\mathcal{K}(t,x,.), f_0\rangle = e^{-2nt}\int_{\mathbb{R}^d} \partial_{x_I}\mathcal{K}(t,x,y)\,f_0(\mathrm{d}y) = e^{-2nt}\,\partial_I\left(e^{tL}f_0\right)(x).$$

The estimate readily comes with the fact that, with the Lebesgue theorem again and the second identity:

$$\begin{aligned}\|\partial_I\left(e^{tL}f_0\right)\|_{\dot{W}^{m-n,1}} \leq \|D_x^m\left(e^{tL}f_0\right)\|_{L^1} &= \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} D_x^m\mathcal{K}(t,x,y)\,f_0(\mathrm{d}y)\right|\mathrm{d}x \\ &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |D_x^n\mathcal{K}(t,x,y)|\,\mathrm{d}x\,|f_0|(\mathrm{d}y) = \left(1-e^{-4t}\right)^{-\frac{n}{2}}\|\gamma^2\|_{\dot{W}^{m,1}}\int_{\mathbb{R}^d}|f_0|(\mathrm{d}y).\end{aligned}$$
□

In particular, for $m = -n+1$, the bound is integrable in time, which shows that integrating in time leads to a better regularity than the source term. It is also not far from being integrable for $m = -n+2$, since we get a bound in t^{-1}, but of course we cannot reach this regularity. However, some kind of cut off in the integral will lead to an interesting bound in order to get as close to this regularity as possible.

Lemma 1.4.9. *Given any $h \in L^\infty\left((0,T), \mathcal{M}^s(\mathbb{R}^d)\right)$ for some $T > 0$, $n \in \mathbb{N}$ and $I \in \{1,...,d\}^n$, there exists a unique solution $f \in \mathcal{C}\left([0,T], W^{-n+1,1}(\mathbb{R}^d)\right)$ to:*

$$\partial_t f = Lf + \partial_I^n h \qquad \text{in } \mathcal{D}'\left((0,T)\times\mathbb{R}^d\right), \qquad\qquad f(0) = 0, \tag{1.4.12}$$

given for all $t \in [0,T]$ by:

$$f(t) = \int_0^t e^{(t-u)L}(\partial_I^n h(u))\,\mathrm{d}u = \partial_I^n\int_0^t e^{-2n(t-u)}e^{(t-u)L}h(u)\,\mathrm{d}u, \tag{1.4.13}$$

where the last integral is to be understood as a Bochner integral. Moreover, some estimates holds:

1. *It satisfies for all $t \in [0,T]$:*

$$\|f(t)\|_{\dot{W}^{-n,1}} \leq e^{-2nt}\int_0^t e^{2nu}\,|h|\big(u,\mathbb{R}^d\big)\,\mathrm{d}u, \tag{1.4.14}$$

$$\begin{aligned}\|f(t)\|_{\dot{W}^{-n+1,1}} &\leq \frac{d}{2}e^{-2nt}\int_0^t \frac{e^{2nu}}{\left(1-e^{-4(t-u)}\right)^{\frac{1}{2}}}\,|h|\big(u,\mathbb{R}^d\big)\,\mathrm{d}u \\ &\leq \frac{d}{2}e^{-2(n-1)t}\int_0^t \frac{e^{-2u}}{\left(1-e^{-4u}\right)^{\frac{1}{2}}}\,\mathrm{d}u\,\left\|e^{2(n-1)u}|h|\big(u,\mathbb{R}^d\big)\right\|_{L_u^\infty}.\end{aligned} \tag{1.4.15}$$

2. *For all $T > t > S > 0$, $f(t) = f_{1,S}(t) + f_{2,S}(t) + f_3(t)$ where $f_{1,S}(t) \in W^{\infty,1}(\mathbb{R}^d)$, $f_{2,S}(t) \in \dot{W}^{-n+1,1}(\mathbb{R}^d)$ and $f_3(t) \in \dot{W}^{-n+3,1}(\mathbb{R}^d)$ are given by*

$$f_{1,S}(t) = \int_0^S e^{-4(t-u)}\, e^{(t-u)L}(\partial_I^n h(u))\,\mathrm{d}u, \qquad f_{2,S}(t) = \int_S^t e^{-4(t-u)}\, e^{(t-u)L}(\partial_I^n h(u))\,\mathrm{d}u,$$

$$f_3(t) = \int_0^t \left(1 - e^{-4(t-u)}\right) e^{(t-u)L}(\partial_I^n h(u))\,\mathrm{d}u,$$

and satisfy:

$$\|f_{1,S}(t)\|_{\dot{W}^{-n+2,1}} \le \frac{\|\gamma^2\|_{\dot{W}^{2,1}}}{2\pi^{\frac{d}{2}}} e^{-2nt}\left[\frac{1}{e^{4t}-e^{4S}} - \frac{1}{e^{4t}-1}\right]^{\frac12}\left(\int_0^S \left(e^{2(n+1)u}\,|h|\big(u,\mathbb{R}^d\big)\right)^2 \mathrm{d}u\right)^{\frac12}, \tag{1.4.16}$$

$$\|f_{2,S}(t)\|_{\dot{W}^{-n+1,1}} \le \frac{d}{4} e^{-2(n+1)t}\left(e^{4t}-e^{4S}\right)^{\frac12}\left\|e^{2nu}\,|h|\big(u,\mathbb{R}^d\big)\right\|_{L^\infty_u}, \tag{1.4.17}$$

$$\|f_3(t)\|_{\dot{W}^{-n+2,1}} \le \frac{\|\gamma^2\|_{\dot{W}^{2,1}}}{2\pi^{\frac{d}{2}}} e^{-2nt}\left(1-e^{-4t}\right)^{\frac12}\left(\int_0^t \left(e^{2(n+1)u}\,|h|\big(u,\mathbb{R}^d\big)\right)^2 \mathrm{d}u\right)^{\frac12}. \tag{1.4.18}$$

3. *If $h \in L^\infty\left((0,T), \mathcal{M}_1^s(\mathbb{R}^d)\right)$, then for all $t \in [0,T]$,*

$$\|x\,f(t)\|_{\dot{W}_x^{-n+1,1}} \le \frac{d}{2} e^{-2nt}\int_0^t \frac{e^{-2u}}{(1-e^{-4u})^{\frac12}}\,\mathrm{d}u\left\|e^{2nu}\int_{\mathbb{R}^d}|x||h|(u,\mathrm{d}x)\right\|_{L^\infty_u} + \left[\frac{\left\||.|\nabla\gamma^2\right\|_{L^1}}{\pi^{\frac{d}{2}}} + n - 1\right] e^{-2nt}\int_0^t e^{2nu}\,|h|\big(u,\mathbb{R}^d\big)\,\mathrm{d}u. \tag{1.4.19}$$

Proof. The first part is easy to prove thanks to the previous remarks and the usual way to deal with the source term thanks to the semigroup of an evolution equation. For the estimates, the first inequality easily follows from (1.4.13) along with the previous estimates:

$$\|f(t)\|_{\dot{W}^{-n,1}} \le \left\|\int_0^t e^{-2n(t-u)} e^{(t-u)L} h(u)\,\mathrm{d}u\right\|_{L^1} \le \int_0^t e^{-2n(t-u)}\left\|e^{(t-u)L}h(u)\right\|_{L^1}\mathrm{d}u \le \int_0^t e^{-2n(t-u)}\,|h|\big(u,\mathbb{R}^d\big)\,\mathrm{d}u.$$

In the same way for the second estimate:

$$\|f(t)\|_{\dot{W}^{-n+1,1}} \le \left\|\nabla\int_0^t e^{-2n(t-u)} e^{(t-u)L} h(u)\,\mathrm{d}u\right\|_{L^1} \le \int_0^t e^{-2n(t-u)}\left\|e^{(t-u)L}h(u)\right\|_{\dot{W}^{1,1}}\mathrm{d}u \le \int_0^t \frac{e^{-2n(t-u)}}{\left(1-e^{-4(t-u)}\right)^{\frac12}}\frac{\|\gamma^2\|_{\dot{W}^{1,1}}}{\pi^{\frac{d}{2}}}\,|h|\big(u,\mathbb{R}^d\big)\,\mathrm{d}u.$$

As for the second part, similar computations may be done, so that for $f_{1,S}$:

$$\begin{aligned}\|f_{1,S}(t)\|_{\dot{W}^{-n+2,1}} &\le \int_0^S \frac{e^{-(4+2n)(t-u)}}{1-e^{-4(t-u)}}\frac{\|\gamma^2\|_{\dot{W}^{2,1}}}{\pi^{\frac{d}{2}}}\,|h|\big(u,\mathbb{R}^d\big)\,\mathrm{d}u\\ &\le \frac{\|\gamma^2\|_{\dot{W}^{2,1}}}{\pi^{\frac{d}{2}}} e^{-2(n+1)t}\int_0^S \frac{e^{-2(t-u)}}{1-e^{-4(t-u)}}\, e^{2(n+1)u}\,|h|\big(u,\mathbb{R}^d\big)\,\mathrm{d}u\\ &\le \frac{\|\gamma^2\|_{\dot{W}^{2,1}}}{\pi^{\frac{d}{2}}} e^{-2(n+1)t}\left(\int_0^S \frac{e^{-4(t-u)}}{(1-e^{-4(t-u)})^2}\,\mathrm{d}u\right)^{\frac12}\left(\int_0^S \left(e^{2(n+1)u}\,|h|\big(u,\mathbb{R}^d\big)\right)^2\mathrm{d}u\right)^{\frac12},\end{aligned}$$

and we find (1.4.16) when we compute

$$\int_0^S \frac{e^{-4(t-u)}}{(1-e^{-4(t-u)})^2}\,\mathrm{d}u = \frac14\left[\frac{1}{1-e^{-4(t-S)}} - \frac{1}{1-e^{-4t}}\right].$$

In the same way for $f_{2,S}$, it yields

$$\|f_{2,S}(t)\|_{\dot{W}^{-n+1,1}} \le \frac{d}{2} e^{-2nt}\int_S^t \frac{e^{-4(t-u)}}{(1-e^{-4(t-u)})^{\frac12}}\, e^{-2nu}\,|h|\big(u,\mathbb{R}^d\big)\,\mathrm{d}u$$

$$\leq \frac{d}{2} e^{-2nt} \int_S^t \frac{e^{-4(t-u)}}{(1-e^{-4(t-u)})^{\frac{1}{2}}} \, \mathrm{d}u \, \left\| e^{2nu} \, |h|\big(u, \mathbb{R}^d\big) \right\|_{L^\infty_u},$$

which is exactly (1.4.17) when we compute the remaining integral. Then, for f_3, it is easy to check that

$$\begin{aligned} \|f_3(t)\|_{\dot{W}^{-n+2,1}} &\leq \int_0^t e^{-(4+2n)(t-u)} \, \frac{\|\gamma^2\|_{\dot{W}^{2,1}}}{\pi^{\frac{d}{2}}} \, |h|\big(u, \mathbb{R}^d\big) \, \mathrm{d}u \\ &\leq \frac{\|\gamma^2\|_{\dot{W}^{2,1}}}{\pi^{\frac{d}{2}}} e^{-2(n+1)t} \left(\int_0^t e^{-4(t-u)} \, \mathrm{d}u \right)^{\frac{1}{2}} \left(\int_0^t \left(e^{2(n+1)u} \, |h|\big(u, \mathbb{R}^d\big) \right)^2 \mathrm{d}u \right)^{\frac{1}{2}}, \end{aligned}$$

and therefore (1.4.18).

The third part is a bit more tricky. For all $t > 0$ we define $f_4(t) \in \dot{W}^{-n+1,1}(\mathbb{R}^d)$ and $f_5(t) \in \dot{W}^{-n+2,1}(\mathbb{R}^d)$ by:

$$f_4(t) = \partial_{\tilde{I}}^{n-1} \left(x \, \partial_{i_n} \int_0^t e^{-2n(t-u)} e^{(t-u)L} h(u) \, \mathrm{d}u \right), \qquad f_5(t) = \left[x, \partial_{\tilde{I}}^{n-1} \right] \partial_{i_n} \int_0^t e^{-2n(t-u)} e^{(t-u)L} h(u) \, \mathrm{d}u,$$

where $\tilde{I} = (i_1, ..., i_{n-1})$ with $I = (i_1, ..., i_n)$. It is obvious that $x\, f(t) = f_4(t) + f_5(t)$. Moreover, $f_5(t)$ is easy to estimate due to the fact that $\left[x, \partial_{\tilde{I}}^{n-1} \right]$ can be readily computed, which leads to:

$$\|f_{5,S}(t)\|_{\dot{W}^{-n+1,1}} \leq (n-1) \left\| \int_0^t e^{-2n(t-u)} e^{(t-u)L} h(u) \, \mathrm{d}u \right\|_{L^1} \leq (n-1) \int_0^t e^{-2n(t-u)} \, |h|\big(u, \mathbb{R}^d\big) \, \mathrm{d}u.$$

For $f_4(t)$, first of all, to get an estimate in $\dot{W}^{-n+1,1}$, we only need to focus on $\tilde{f}_4(t)$ in L^1 where

$$\tilde{f}_4(t) = x \, \partial_{i_n} \int_0^t e^{-2n(t-u)} e^{(t-u)L} h(u) \, \mathrm{d}u,$$

since $f_4(t) = \partial_{\tilde{I}}^{n-1} \tilde{f}_4(t)$. For this, we will estimate

$$f_6(t) = x \int_{\mathbb{R}^d} \partial_{x_i} \mathcal{K}(t,x,y) \, f_0(\mathrm{d}y) = \int_{\mathbb{R}^d} (x - e^{-2t} y) \, \partial_{x_i} \mathcal{K}(t,x,y) \, f_0(\mathrm{d}y) + e^{-2t} \int_{\mathbb{R}^d} \partial_{x_i} \mathcal{K}(t,x,y) \, y \, f_0(\mathrm{d}y).$$

Using the expression of $\mathcal{K}$, the first term on the right-hand side can be estimated thanks to (1.4.11):

$$\left\| \int_{\mathbb{R}^d} (x - e^{-2t} y) \, \partial_{x_i} \mathcal{K}(t,x,y) \, f_0(\mathrm{d}y) \right\|_{L^1} \leq \pi^{-\frac{d}{2}} \, |f_0|(\mathbb{R}^d) \int |x| |\nabla \gamma^2|.$$

The second term is also easy to estimate:

$$\left\| \int_{\mathbb{R}^d} \partial_{x_i} \mathcal{K}(t,x,y) \, y \, f_0(\mathrm{d}y) \right\|_{L^1} \leq \frac{\left(1 - e^{-4t}\right)^{-\frac{1}{2}}}{\pi^{\frac{d}{2}}} \|\gamma^2\|_{\dot{W}^{1,1}} \int_{\mathbb{R}^d} |y| \, |f_0|(\mathrm{d}y).$$

Coming back to $\tilde{f}_4(t)$, those estimates lead to

$$\begin{aligned} \|\tilde{f}_4(t)\|_{L^1} &\leq \pi^{-\frac{d}{2}} \int_0^t e^{-2n(t-u)} \left[|h|(u, \mathbb{R}^d) \, \||.| \nabla \gamma^2\|_{L^1} + \frac{e^{-2(t-u)}}{(1-e^{-4(t-u)})^{\frac{1}{2}}} \|\gamma^2\|_{\dot{W}^{1,1}} \int_{\mathbb{R}^d} |y| \, |h|(u, \mathrm{d}y) \right] \mathrm{d}u \\ &\leq \frac{\||.| \nabla \gamma^2\|_{L^1}}{\pi^{\frac{d}{2}}} e^{-2nt} \int_0^t e^{2nu} |h|(u, \mathbb{R}^d) \, \mathrm{d}u + \frac{d}{2} e^{-2nt} \int_0^t \frac{e^{-2(t-u)}}{(1-e^{-4(t-u)})^{\frac{1}{2}}} \, e^{2nu} \int_{\mathbb{R}^d} |y| \, |h|(u, \mathrm{d}y) \, \mathrm{d}u \\ &\leq \frac{\||.| \nabla \gamma^2\|_{L^1}}{\pi^{\frac{d}{2}}} e^{-2nt} \int_0^t e^{2nu} |h|(u, \mathbb{R}^d) \, \mathrm{d}u + \frac{d}{2} e^{-2nt} \int_0^t \frac{e^{-2v}}{(1-e^{-4v})^{\frac{1}{2}}} \, \mathrm{d}v \left\| e^{2nu} \int_{\mathbb{R}^d} |y| \, |h|(u, \mathrm{d}y) \right\|_{L^\infty_u}, \end{aligned}$$

which is exactly what we need to get (1.4.19) when putting all back together. □

We dealt with the spatial derivative in the source term. Actually, thanks to the linearity of this equation, we can also deal with a time-derivative, subject to a slightly higher regularity for the source term.

Corollary 1.4.10. *Let $h \in L^\infty\left((0,T), \mathcal{M}_1(\mathbb{R}^d)\right) \cap \mathcal{C}\left([0,T), \mathcal{M}(\mathbb{R}^d)\right)$ for some $T > 0$, $n \in \mathbb{N}$ and $I \in \{1, ..., d\}^n$. Then there exists a unique solution $f \in L^\infty\left((0,T), W^{-n-1,1}(\mathbb{R}^d)\right)$ of the Fokker-Planck equation with source term:*

$$\partial_t f = L f + \partial_t \partial_I^n h \qquad \text{in } \mathcal{D}'\left((0,T) \times \mathbb{R}^d\right), \qquad\qquad f(0) = 0.$$

It is given by the identity $f = \Delta g + \nabla \cdot (2y\, g) + \partial_I^n h - e^{-2nt} \partial_I^n (e^{tL} h(0))$ where g is the unique solution in $\mathcal{C}\left((0,T), W^{-n+1,1}(\mathbb{R}^d)\right)$ to (1.4.12)

Proof. Suppose that such an f exists. Define

$$g(t) = \int_0^t \left(f(u) + e^{uL}(\partial_I^n h(0))\right) \mathrm{d}u \in W^{1,\infty}\left((0,T), W^{-n-1,1}(\mathbb{R}^d)\right),$$

so that $\partial_t g = f + e^{uL}(\partial_I^n h(0))$ and thus:

$$\partial_t \left(\partial_t g - Lg - \partial_I^n h\right) = 0, \qquad g(0) = 0.$$

Moreover, $\partial_t g(0) = \partial_I^n h(0)$ and $Lg(0) = 0$ so that

$$\partial_t g - Lg - \partial_I^n h = 0, \qquad g(0) = 0.$$

The result obviously follows. □

1.4.3. Proof of Lemma 1.4.3

Duality and regularization. Lemma 1.4.9 provides interesting estimates in view of Lemma 1.4.3. Indeed, there are many source terms in (1.4.6) with different regularities, and we can apply for each of them one of the previous estimates with different n by linearity of the Fokker-Planck operator L. However, it is obvious that we will not be able to reach (at least at first) a non-negative regularity for all the estimates, for instance because of the $\tau(t)^{-2}\,\Delta^2 g_3$ term for which we have $n = 4$ in Lemma 1.4.9, and the best estimate we can get is for $\dot{W}^{-3,1}$. Therefore, if we want to estimate in a higher regularity, we need to use duality and regularize the test function to fit the lower regularity for which we have the estimate (for instance with a convolution). We also need to check if this regularization suits the estimate, i.e. if we can get a nice convergence rate for the difference between the initial test function and the regularized one in L^∞ thanks to the assumption that $f(t)$ is in L^1 uniformly in t. For example, if one wants to have a convergence rate in L^1 strong through this way, they would have to regularize an L^∞ test function into a smoother function. However, approaching a general L^∞-function by a regular function is not very convenient (if not doomed). Actually, there is a more suitable case: the Wasserstein (or Kantorovich–Rubinstein) distance $\mathcal{W}_1$. Indeed, such a distance has a dual representation: for any μ_1, μ_2 in $\mathcal{P}_1(\mathbb{R}^d)$,

$$\mathcal{W}_1(\mu_1, \mu_2) = \sup\left\{\int_{\mathbb{R}^d} \Phi\,\mathrm{d}(\mu_1 - \mu_2), \Phi : \mathbb{R}^d \to \mathbb{R} \text{ continuous}, \mathrm{Lip}(\Phi) \leq 1\right\}.$$

The fact that Φ is 1-Lip is suitable in order to regularize it, whereas the fact that Φ may be unbounded (but growing at most like an affine function) is not a big problem thanks to the assumption on the integrability of f (in particular its uniformly bounded second momentum).

Given any $\Phi : \mathbb{R}^d \to \mathbb{R}$ 1-Lip, before using the estimates in Lemma 1.4.9, we need to quantify the cost of its regularization into a smoother function. We will regularize it into a $\mathcal{C}_c^\infty$ function since it is not very difficult. Our first action is to regularize Φ into a $\mathcal{C}^\infty$ function by convolution with a smooth and suitable mollifier. Take some $\Psi \in \mathcal{S}(\mathbb{R}^d)$ such that $\Psi \geq 0$ and $\int_{\mathbb{R}^d} \Psi = 1$. For $\delta > 0$, define Ψ^δ by $\Psi^\delta(x) = \frac{1}{\delta^d}\Psi\left(\frac{x}{\delta}\right)$ for all $x \in \mathbb{R}^d$. Then it is known that $\tilde{\Phi}^\delta := \Phi * \Psi^\delta$ is a $\mathcal{C}^\infty$ function and satisfies:

1. $\|\tilde{\Phi}^\delta - \Phi\|_{L^\infty} \leq \delta\,\|\,|.|\,\Psi\|_{L^1}$,
2. $\mathrm{Lip}(\tilde{\Phi}^\delta) \leq 1$ and in a more general way, $\forall n \in \mathbb{N}$, $\|\tilde{\Phi}^\delta\|_{\dot{W}^{1+n,\infty}} \leq \delta^{-n}\,\|\Psi\|_{\dot{W}^{n,\infty}}$.

In particular, the first estimate yields

$$\left|\int_{\mathbb{R}^d} \tilde{\Phi}^\delta\,\mathrm{d}(f(t) - \pi^{-\frac{d}{2}}\gamma^2) - \int_{\mathbb{R}^d} \Phi\,\mathrm{d}(f(t) - \pi^{-\frac{d}{2}}\gamma^2)\right| \leq 2\delta\,\|\,|.|\,\Psi\|_{L^1(\mathbb{R}^d)}.$$

Now, we want to apply a cut off to the function $\tilde{\Phi}^\delta$. It is in this step where the fact that the second momentum of $f(t)$ is bounded independently of t is used. Take $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that $\chi \equiv 1$ on $\mathcal{B}(0,1)$ and $0 \leq \chi \leq 1$. Define χ^δ by $\chi^\delta(x) = \chi(\delta x)$. Then $\Phi^\delta := (\tilde{\Phi}^\delta - \tilde{\Phi}^\delta(0))\chi^\delta \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and we get some similar properties from simple computations:

1. Given any $x \in \mathbb{R}^d \setminus \{0\}$, we get

$$\left|\frac{(\tilde{\Phi}^\delta(x) - \tilde{\Phi}^\delta(0)) - \Phi^\delta(x)}{|x|^2}\right| = \left|\frac{(\tilde{\Phi}^\delta(x) - \tilde{\Phi}^\delta(0))(1 - \chi^\delta(x))}{|x|^2}\right| = \frac{|\tilde{\Phi}^\delta(x) - \tilde{\Phi}^\delta(0)|}{|x|}\,\frac{|1 - \chi(\delta x)|}{|x|}$$
$$\leq \frac{|1 - \chi(\delta x)|}{|x|} \leq \delta,$$

so that

$$\left\|\frac{(\tilde{\Phi}^\delta - \tilde{\Phi}^\delta(0)) - \Phi^\delta}{|.|^2}\right\|_{L^\infty} \leq \delta. \tag{1.4.20}$$

2. We also get

$$\nabla\Phi^\delta(x) = \nabla\left((\Phi-\Phi(0))\,\chi^{2,\delta}\right)(x) = \nabla\Phi(x)\,\chi^{2,\delta}(x) + (\Phi(x)-\Phi(0))\,\nabla\chi^{2,\delta}(x)$$
$$= \nabla\Phi(x)\,\chi(\delta x) + \delta(\Phi(x)-\Phi(0))\,\nabla\chi(\delta x),$$

and therefore

$$\left|\nabla\left((\Phi-\Phi(0))\,\chi^{2,\delta}\right)(x)\right| \leq \chi(\delta x) + \delta|x|\;|\nabla\chi(\delta x)|$$
$$\leq \|\chi + |.|\,|\nabla\chi|\,\|_{L^\infty}.$$

Hence, Φ^δ is Lip uniformly in δ:

$$\mathrm{Lip}\left(\Phi^\delta\right) \leq \|\chi + |.|\,|\nabla\chi|\,\|_{L^\infty}.$$

3. In the same way, computing the n-th derivative of Φ^δ leads to the following property:

$$\forall n\in\mathbb{N}, \exists C_n>0, \forall\delta\in(0,1], \|\Phi^\delta\|_{\dot W^{1+n,\infty}} \leq C_n\,\delta^{-n}. \tag{1.4.21}$$

In particular, given ant $t\geq 0$, (1.4.20) along with the fact that $\int_{\mathbb{R}^d}\tilde\Phi^\delta(0)\,\mathrm{d}(f(t)-\gamma^2) = 0$ lead to

$$\left|\int_{\mathbb{R}^d}\Phi^\delta\,\mathrm{d}(f(t)-\gamma^2) - \int_{\mathbb{R}^d}\tilde\Phi^\delta\,\mathrm{d}(f(t)-\gamma^2)\right| \leq \delta\int_{\mathbb{R}^d}|y|^2\,|f(t)-\gamma^2|(\mathrm{d}y).$$

Therefore, the cost of the regularization of Φ into Φ^δ is only proportional to δ. In view of the convergence rate which must be reached (in e^{-2t}), we should define $\delta(t) = e^{-2t}$ and consider $\Phi^{\delta(t)}$. Therefore, the previous estimates yield

$$\left|\int_{\mathbb{R}^d}\Phi^{\delta(t)}\,(f(t)-\pi^{-\frac{d}{2}}\,\gamma^2)\,\mathrm{d}y - \int_{\mathbb{R}^d}\Phi\,(f(t)-\pi^{-\frac{d}{2}}\,\gamma^2)\,\mathrm{d}y\right| \leq (C_0 + \|\,|y|^2 f\,\|_{L^\infty_t L^1_y})\,e^{-2t}. \tag{1.4.22}$$

End of the proof. It remains to estimate $\int_{\mathbb{R}^d}\Phi^{\delta(t)}\,(f(t)-\pi^{-\frac{d}{2}}\gamma^2)\,\mathrm{d}y$. We now use the fact that f satisfies (1.4.6). Define f_1, f_2, f_3 and f_4 to be the solutions to (1.4.12) respectively with source terms $e^{-2t}\,\nabla\cdot g_1$, $e^{-2t}\,\nabla\cdot g_2$, $\Delta^2 g_3$ and $\nabla\cdot(\nabla\cdot g_4)$. Define also $f_0 = e^{tL} f_{\mathrm{in}}$. Therefore, f may be written in terms of f_0, f_1, f_2, f_3 and f_4 thanks to Lemma 1.4.9 and Corollary 1.4.10:

$$f = f_0 + f_1 + f_3 + f_4 + \Delta f_2 + \nabla\cdot(2y\,f_2) + e^{-2t}\,\nabla\cdot g_2 - e^{-2t}\,\nabla\cdot(e^{tL}g_2(0)). \tag{1.4.23}$$

— For f_0, thanks to Lemma 1.4.1 and to the inequality $\mathcal{W}_1\leq\mathcal{W}_2$, it is known that

$$\mathcal{W}_1\left(f_0(t),\pi^{-\frac{d}{2}}\,\gamma^2\right) \leq e^{-2t}\,\mathcal{W}_2\left(f_{\mathrm{in}},\pi^{-\frac{d}{2}}\,\gamma^2\right) \leq C_0\,e^{-2t}\,(1+\|\,|y|^2 f_{\mathrm{in}}\,\|_{L^1}), \qquad \forall t\geq 0.$$

Moreover, $\|\,|y|^2 f_{\mathrm{in}}\,\|_{L^1} \leq \|\,|y|^2 f\,\|_{L^\infty_t L^1_y}$ thanks to the assumption $f\in\mathcal{C}((0,\infty),L^1_w(\mathbb{R}^d))$. Therefore, there holds for all $t\geq 0$:

$$\left|\int_{\mathbb{R}^d}\Phi^{\delta(t)}\,(f_0(t)-\pi^{-\frac{d}{2}}\gamma^2)\,\mathrm{d}y\right| \leq \mathrm{Lip}(\Phi^{\delta(t)})\,\pi^{\frac{d}{2}}\,\mathcal{W}_1\left(f_0(t),\pi^{-\frac{d}{2}}\gamma^2\right) \leq C_0\,(1+\|\,|y|^2 f\,\|_{L^\infty_t L^1_y})\,e^{-2t}.$$

Thus, it suffices to prove that any of the other terms in (1.4.23) integrated against $\Phi^{\delta(t)}$ goes to 0 with the same exponential convergence rate.

— For the last two terms, this convergence is easy to state:

$$\langle\Phi^{\delta(t)}, e^{-2t}\,\nabla\cdot(e^{tL}g_2(0))\rangle = -e^{-2t}\int_{\mathbb{R}^d}\nabla\Phi^{\delta(t)}\,e^{tL}g_2(0)\,\mathrm{d}y,$$
$$\left|\langle\Phi^{\delta(t)}, e^{-2t}\,\nabla\cdot(e^{tL}g_2(0))\rangle\right| \leq e^{-2t}\|\nabla\Phi^{\delta(t)}\|_{L^\infty}\,\|e^{tL}g_2(0)\|_{L^1} \leq C_0\,|g_2|(0,\mathbb{R}^d)\,e^{-2t} \leq C_0\,G\,e^{-2t},$$

and in the same way

$$\left|\langle\Phi^{\delta(t)}, e^{-2t}\,\nabla\cdot g_2(t)\rangle\right| \leq e^{-2t}\|\nabla\Phi^{\delta(t)}\|_{L^\infty}\,|g_2|(t,\mathbb{R}^d) \leq C_0\,G\,e^{-2t}.$$

— For f_1, we use (1.4.14) with $n=1$ to get for all $t\geq 0$:

$$\|f_1(t)\|_{\dot W^{-1,1}} \leq e^{-2t}\int_0^t|g_1|(u,\mathbb{R}^d)\,\mathrm{d}u \leq e^{-2t}\left(\int_0^t\left(e^{2u}\,|g_1|(u,\mathbb{R}^d)\right)^2\mathrm{d}u\right)^{\frac{1}{2}}\left(\int_0^t e^{-4u}\,\mathrm{d}u\right)^{\frac{1}{2}} \leq \frac{G}{2}\,e^{-2t}.$$

Therefore, for all $t\geq 0$,

$$\left|\langle\Phi^{\delta(t)}, f_1(t)\rangle\right| \leq \|\Phi^{\delta(t)}\|_{\dot W^{1,\infty}}\,\|f_1(t)\|_{\dot W^{-1,1}} \leq C_0\,G\,e^{-2t}.$$

— For f_4, we use again (1.4.14) with $n = 2$:

$$\|f_4(t)\|_{\dot W^{-n,1}} \le e^{-4t} \int_0^t e^{4u}\, |g_4|\big(u, \mathbb{R}^d\big)\, \mathrm{d}u \le G\, e^{-4t}.$$

We conclude for this term using the fact that $\|\Phi^{\delta(t)}\|_{\dot W^{2,\infty}} \le C_0\, e^{2t}$ thanks to (1.4.21):

$$\Big|\langle \Phi^{\delta(t)}, f_4(t)\rangle\Big| \le \|\Phi^{\delta(t)}\|_{\dot W^{2,\infty}}\, \|f_4(t)\|_{\dot W^{-2,1}} \le C_0\, G\, e^{-2t}.$$

— For f_3, we use the second inequality in (1.4.15) with $n = 4$ along with the fact that $e^{-2u}(1 - e^{-4u})^{-\frac{1}{2}}$ is integrable on $(0, \infty)$:

$$\|f_3(t)\|_{\dot W^{-3,1}} \le C_0\, e^{-6t}\, \big\|e^{6u}\, |g_3|\big(u, \mathbb{R}^d\big)\big\|_{L^\infty_u} \le C_0\, G\, e^{-6t}, \qquad \forall t \ge 0.$$

Property (1.4.21) shows that $\|\phi^{\delta(t)}\|_{\dot W^{3,\infty}} \le C_0\, e^{4t}$, and thus

$$\Big|\langle \Phi^{\delta(t)}, f_3(t)\rangle\Big| \le \|\phi^{\delta(t)}\|_{\dot W^{3,\infty}}\, \|f_3(t)\|_{\dot W^{-3,1}} \le C_0\, G\, e^{-2t}.$$

— As for $\nabla \cdot (2y f_2)$, we use (1.4.19) with $n = 1$, so that for all $t \ge 0$:

$$\|\nabla \cdot (2y f_2)\|_{\dot W^{-1,1}} = 2\||y| f_2\|_{L^1} \le C_0\, e^{-2t} \left[\left\| \int_{\mathbb{R}^d} |x| |g_2|(u, \mathrm{d}x) \right\|_{L^\infty_u} + \int_0^t |g_2|\big(u, \mathbb{R}^d\big) \right] \le C_0\, G\, e^{-2t},$$

thanks to the fact that

$$\int_0^t |g_2|\big(u, \mathbb{R}^d\big) \le \left(\int_0^t \big(e^{2u}\, |g_2|\big(u, \mathbb{R}^d\big)\big)^2 \mathrm{d}u \right)^{\frac{1}{2}} \left(\int_0^t e^{-4u}\, \mathrm{d}u \right)^{\frac{1}{2}} \le C_0\, G.$$

Thus,

$$\Big|\langle \Phi^{\delta(t)}, \nabla \cdot (2y f_2(t))\rangle\Big| \le \|\Phi^{\delta(t)}\|_{\dot W^{1,\infty}}\, \|\nabla \cdot (2y f_2(t))\|_{\dot W^{-1,1}} \le C_0\, G\, e^{-2t}.$$

— Lastly, we will use the decomposition used in the part 2 of Lemma 1.4.9 for Δf_2: for some $S > 0$, $f_2(t) = f_2^{1,S}(t) + f_2^{2,S}(t) + f_2^3(t)$ and with (1.4.16)-(1.4.18) for $n = 1$,

$$\|\Delta f_2^{1,S}(t)\|_{\dot W^{-1,1}} = \|f_2^{1,S}(t)\|_{\dot W^{1,1}} \le C_0\, e^{-2t} \left[\frac{1}{e^{4t} - e^{4S}} - \frac{1}{e^{4t} - 1} \right]^{\frac{1}{2}} \left(\int_0^S \big(e^{2u}\, |g_2|\big(u, \mathbb{R}^d\big)\big)^2 \mathrm{d}u \right)^{\frac{1}{2}},$$

$$\|\Delta f_2^{2,S}(t)\|_{\dot W^{-2,1}} = \|f_2^{2,S}(t)\|_{L^1} \le C_0\, e^{-4t}\, \big(e^{4t} - e^{4S}\big)^{\frac{1}{2}}\, \big\||g_2|\big(u, \mathbb{R}^d\big)\big\|_{L^\infty_u},$$

$$\|\Delta f_2^3(t)\|_{\dot W^{-1,1}} = \|f_2^3(t)\|_{\dot W^{1,1}} \le C_0\, e^{-2t} \left(\int_0^t \big(e^{2u}\, |g_2|\big(u, \mathbb{R}^d\big)\big)^2 \mathrm{d}u \right)^{\frac{1}{2}}.$$

Therefore, those estimates yield

$$\begin{aligned} \Big|\langle \Phi^{\delta(t)}, \Delta f_2(t)\rangle\Big| &\le \|\Phi^{\delta(t)}\|_{\dot W^{1,\infty}} \Big[\|\Delta f_2^{1,S}(t)\|_{\dot W^{-1,1}} + \|\Delta f_2^{3,S}(t)\|_{\dot W^{-1,1}} \Big] + \|\Phi^{\delta(t)}\|_{\dot W^{2,\infty}}\, \|\Delta f_2^{2,S}(t)\|_{\dot W^{-2,1}} \\ &\le C_0\, G\, e^{-2t} \Big[(e^{4t} - e^{4S})^{-\frac{1}{2}} + (e^{4t} - e^{4S})^{\frac{1}{2}} \Big]. \end{aligned}$$

The convergence rate comes by optimizing over S, which is taking $S = S(t)$ such that $e^{4t} - e^{4S} = 1$.

Putting all together, we finally get that

$$\left| \int_{\mathbb{R}^d} \Phi^{\delta(t)}\, \big(f(t) - \pi^{-\frac{d}{2}} \gamma^2\big)\, \mathrm{d}y \right| \le C_0\, \big(1 + G + \|\, |y|^2 f\, \|_{L^\infty_t L^1_y}\big)\, e^{-2t},$$

and the result by putting it back in (1.4.22). □

1.4.4. Proof of Corollaries 1.1.8 and 1.1.14

The only thing that remains is the proof of Corollaries 1.1.8 and 1.1.14. Like already said, the convergence rate in $(1+\delta)$-Wasserstein distance for $\delta \in (0,1)$ (in both Corollaries) follows from a simple Hölder inequality and the bounds of the second momentum of both $|v_\varepsilon(t)|^2$ and $\tilde{\rho}(t)$ found in Theorem 1.1.5 and 1.1.10 respectively. On the other hand, the convergence rate in $W^{-1+\delta,1}$ can be proved through the following lemma and the inequality $\| \, \|_{\dot{W}^{-1,1}} \leq \mathcal{W}_1$.

Lemma 1.4.11. *Given any $\delta \in (0,1)$ and any $f \in L^1(\mathbb{R}^d)$, there holds*

$$\|f\|_{\dot{W}^{-1+\delta,1}} \leq C_0 \, \|f\|_{\dot{W}^{-1,1}}^{1-\delta} \, \|f\|_{L^1}^{\delta}.$$

Proof. Let $g \in \dot{W}^{1-\delta,\infty}(\mathbb{R}^d) = \mathcal{C}^{0,1-\delta}(\mathbb{R}^d)$ and define $g_\eta = g * \gamma_\eta$ where $\gamma_\eta = \gamma_\eta(x) = (\pi\eta)^{-\frac{d}{2}} e^{-\frac{|x|^2}{\eta}}$ for all $\eta > 0$. Then, for any $x \in \mathbb{R}^d$,

$$|g(x) - g_\eta(x)| \leq \int_{\mathbb{R}^d} |g(x) - g(x-y)| \, \gamma_\eta(y) \, \mathrm{d}y \leq \int_{\mathbb{R}^d} \|g\|_{\mathcal{C}^{0,1-\delta}} \, |y|^{1-\delta} \, \gamma_\eta(y) \, \mathrm{d}y \leq \eta^{1-\delta} \, \|g\|_{\mathcal{C}^{0,1-\delta}} \| \, |.|^{1-\delta} \, \gamma \, \|_{L^1}.$$

Moreover, $\nabla g_\eta = g * \nabla \gamma_\eta$, so that for any $x \in \mathbb{R}^d$,

$$|\nabla g_\eta(x)| = \left| \int_{\mathbb{R}^d} (g(x-y) - g(x)) \nabla \gamma_\eta(y) \, \mathrm{d}y \right| \leq \|g\|_{\mathcal{C}^{0,1-\delta}} \int_{\mathbb{R}^d} |y|^{1-\delta} \, |\nabla \gamma_\eta(y)| \, \mathrm{d}y \leq \eta^{-\delta} \, \|g\|_{\mathcal{C}^{0,1-\delta}} \| \, |.|^{1-\delta} \nabla \gamma \, \|_{L^1}.$$

Therefore, we get for all $\eta > 0$:

$$\begin{aligned} \left| \int_{\mathbb{R}^d} f(x) \, g(x) \, \mathrm{d}x \right| &\leq \left| \int_{\mathbb{R}^d} f(x) \, (g(x) - g_\eta(x)) \, \mathrm{d}x \right| + \left| \int_{\mathbb{R}^d} f(x) \, g_\eta(x) \, \mathrm{d}x \right| \\ &\leq \|f\|_{L^1} \, \|g - g_\eta\|_{L^\infty} + \|f\|_{\dot{W}^{-1,1}} \, \|\nabla g_\eta\|_{L^\infty} \leq C_0 \, \|g\|_{\dot{W}^{1-\delta,\infty}} (\eta^{1-\delta} \, \|f\|_{L^1} + \eta^{-\delta} \, \|f\|_{\dot{W}^{-1,1}}), \end{aligned}$$

which yields

$$\|f\|_{\dot{W}^{-1+\delta,1}} \leq C_0 \, (\eta^{1-\delta} \, \|f\|_{L^1} + \eta^{-\delta} \, \|f\|_{\dot{W}^{-1,1}}),$$

and the result by optimizing in η. □

1.5. Kinetic Isothermal Euler system

1.5.1. Discussion on its formal properties

We recall the Kinetic Isothermal Euler system (1.1.16):

$$\partial_t f + \xi \cdot \nabla_x f - \lambda \, \nabla_x (\ln \rho) \cdot \nabla_\xi f = 0,$$

where $\lambda > 0$ and $\rho(t,x) = \int f(t,x,\mathrm{d}\xi)$. A solution $f = f(t,x,\xi)$ of such a Vlasov equation should be a non-negative measure in x and ξ for every (or a.e.) t.

This equation is a non-linear Vlasov-type equation with potential $\ln \rho$. In particular, it is a transport equation with null-divergence transport. The formal properties of this kind of equations should be guaranteed, i.e. the conservation of the mass and the energy like for the Schrödinger equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} f(t, \mathrm{d}x, \mathrm{d}\xi) \right) = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\xi|^2 \, f(t, \mathrm{d}x, \mathrm{d}\xi) + \lambda \int_{\mathbb{R}^d} \rho(t,x) \ln \rho(t,x) \, \mathrm{d}x \right) = 0.$$

The second equation is very interesting. Indeed, it transforms the highly singular non-linearity of the equation (1.1.16) $\nabla_x (\ln \rho)$ into better suited properties on ρ. Moreover, if we want $\int_{\mathbb{R}^d} \rho(t) \ln \rho(t) \, \mathrm{d}x$ to be well-defined, we shall require ρ to be in $L \log L$ and in particular in L^1, which is similar to the previous properties found for the Wigner Measure. Furthermore, we should also have some other (formal) properties coming from (formal) computations, for example for ρ or also for $J(t,x) := \int_{\mathbb{R}^d} \xi \, f(t,x,\mathrm{d}\xi)$:

$$\partial_t \rho(t,x) + \nabla_x \cdot J(t,x) = 0, \tag{1.5.1}$$

$$\partial_t J(t,x) + \nabla_x \cdot \int_{\mathbb{R}^d} \xi \otimes \xi \, f(t,x,\mathrm{d}\xi) + \lambda \, \nabla_x \rho(t,x) = 0. \tag{1.5.2}$$

Those two equations look like (1.1.17). In particular, if we consider time-dependent mono-kinetic solutions to (1.1.16), then (1.5.1) and (1.5.2) give exactly (1.1.17). Furthermore, they yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}|x|^2\,f(t,\mathrm{d}x,\mathrm{d}\xi)\right)=2\iint_{\mathbb{R}^d\times\mathbb{R}^d}x\cdot\xi\,f(t,\mathrm{d}x,\mathrm{d}\xi),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}x\cdot\xi\,f(t,\mathrm{d}x,\mathrm{d}\xi)\right)=\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\xi|^2\,f(t,\mathrm{d}x,\mathrm{d}\xi)+\lambda\int_{\mathbb{R}^d}\rho(t,x)\,\mathrm{d}x.$$

All those properties are totally formal. However, a good framework for (1.1.16) should get those properties, which means that all those terms should be well-defined (in some sense). Thus, intuitively, the solution f should be at least in $L^\infty_{\mathrm{loc}}((0,\infty),\mathcal{M}\Sigma_{\log}\cap\mathcal{M}_2)$ where:

$$\mathcal{M}\Sigma_{\log}=\left\{\mu\in\mathcal{M}(\mathbb{R}^d_x\times\mathbb{R}^d_\xi),\int_{\mathbb{R}^d_\xi}\mu(x,\mathrm{d}\xi)\in L^1\cap L\,\log L\,(\mathbb{R}^d)\right\},$$
$$\mathcal{M}_2=\left\{\mu\in\mathcal{M}(\mathbb{R}^d_x\times\mathbb{R}^d_\xi),\iint_{\mathbb{R}^d\times\mathbb{R}^d}(|x|^2+|\xi|^2)\,\mathrm{d}\mu<\infty\right\}.$$

Again, from (1.5.1) and (1.5.2), we can also prove some continuity for ρ and J. Indeed, (1.5.1) implies that $\partial_t\rho\in L^\infty_{\mathrm{loc}}\left((0,\infty),W^{-1-\delta,1}(\mathbb{R}^d)\right)$ for all $\delta>0$ uniformly in δ. Since $\rho\in L^\infty_{\mathrm{loc}}\left((0,\infty),L^1_2\cap L\,\log L(\mathbb{R}^d)\right)$, the previous property leads to $\rho\in W^{1,\infty}_{\mathrm{loc}}\left((0,\infty),W^{-1,1}(\mathbb{R}^d)\right)$ and also $\pi^{-\frac{d}{2}}\,\rho\in\mathcal{C}\left((0,\infty),\mathcal{P}_1(\mathbb{R}^d)\right)$. As for J, similar arguments as in Remark 1.3.7 apply and lead to $J\in\mathcal{C}^{0,1}_{\mathrm{loc}}\left((0,\infty),W^{-1,1}(\mathbb{R}^d)^d\right)\cap\mathcal{C}\left((0,\infty),\mathcal{M}^s(\mathbb{R}^d)^d\right)$.

Actually, (1.5.1) and (1.5.2) are very similar to (1.3.4). Moreover, we also have conservation of the mass and the energy similar to those for the logarithmic Schrödinger equation. Finally, we have seen that the rescaling (1.1.5) is translated into the identity (1.1.12), therefore it is natural to consider a rescaling for the solution of the Kinetic Isothermal Euler system to $\tilde f=\tilde f(t,y,\eta)$ defined by:

$$f(t,x,\xi)=\frac{f_0(\mathbb{R}^d\times\mathbb{R}^d)}{\|\gamma^2\|_{L^1}}\,\tilde f\left(t,\frac{x}{\tau(t)},\tau(t)\,\xi-\dot\tau(t)\,x\right).$$

Thus, we can perform arguments similar to that in the proof of Theorem 1.1.5:

— We define the density of particles and the density of angular momentum:

$$\tilde\rho(t,y):=\int_{\mathbb{R}^d}\tilde f(t,y,\mathrm{d}\eta)\in L^\infty_{loc}\left((0,\infty),L^1_2\cap L\,\log L\,(\mathbb{R}^d)\right),$$
$$\tilde J(t,y):=\int_{\mathbb{R}^d}\eta\,\tilde f(t,y,\mathrm{d}\eta)\in L^\infty_{loc}\left((0,\infty),\mathcal{M}^s_1(\mathbb{R}^d)^d\right),$$

where $\mathcal{M}^s_1(\mathbb{R}^d)$ is the set of signed finite measure with bounded first momentum.

— We also define the modified kinetic energy, the relative entropy and the modified total energy:

$$\mathcal{E}_{\mathrm{kin}}(t):=\frac{1}{2\,\tau(t)^2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\eta|^2\,\tilde f(t,\mathrm{d}y,\mathrm{d}\eta),\qquad\mathcal{E}_{\mathrm{ent}}(t):=\int_{\mathbb{R}^d}\tilde\rho(t,y)\ln\frac{\rho(t,y)}{\gamma^2(y)}\,\mathrm{d}y,$$
$$\mathcal{E}:=\mathcal{E}_{\mathrm{kin}}+\lambda\,\mathcal{E}_{\mathrm{ent}}.$$

— Then, in the same way as in Remark 1.3.5, there holds

$$\dot{\mathcal{E}}=-2\frac{\dot\tau(t)}{\tau(t)}\,\mathcal{E}_{\mathrm{kin}},$$
$$\partial_t\tilde\rho+\frac{1}{\tau^2(t)}\nabla\cdot\tilde J=0,\qquad\partial_t\tilde J+\lambda\,\nabla\tilde\rho+2\lambda\,y\,\tilde\rho=-\frac{1}{\tau^2(t)}\nabla\cdot\int_{\mathbb{R}^d}\eta\otimes\eta\,\tilde f(t,y,\mathrm{d}\eta),\qquad\text{in }\mathcal{D}'.\tag{1.5.3}$$

— Write

$$\mathcal{E}_+:=\mathcal{E}_{\mathrm{kin}}+\lambda\int_{\tilde\rho>1}\tilde\rho\ln\tilde\rho+\lambda\int|y|^2\,\tilde\rho\geq 0,\qquad\mathcal{E}_-:=-\lambda\int_{\tilde\rho<1}\tilde\rho\ln\tilde\rho\geq 0,$$

so that

$$\mathcal{E}=\mathcal{E}_+-\mathcal{E}_-\leq\mathcal{E}(0),\qquad\mathcal{E}_-\leq C_0\,\left(\mathcal{E}_+\right)^{\frac{d}{2(d+2)}}.$$

Similar arguments as in Lemma 1.3.1 apply to this case, showing that $\mathcal{E}_+$ is bounded which leads to the estimates

$$\int_{\mathbb{R}^d} \left(1+|y|^2+|\ln\tilde{\rho}|\right)\tilde{\rho}\,\mathrm{d}y + \frac{1}{\tau(t)^2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\eta|^2\,\tilde{f}(t,\mathrm{d}y,\mathrm{d}\eta) \leq C_0, \qquad \forall t\geq 0,$$
$$\int_0^\infty \frac{\dot{\tau}(t)}{\tau(t)^3}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\eta|^2\,\tilde{f}(t,\mathrm{d}y,\mathrm{d}\eta)\leq C_0.$$

— Those estimates along with the system (1.5.3) show that we can apply Lemma 1.4.4 with (up to a factor $\pi^{-\frac{d}{2}}$) $f=\tilde{\rho}$, $h_1=\tau(t)^{-1}\tilde{J}$, $h_2=0$ and $h_3=\frac{1}{\tau(t)^2}\int_{\mathbb{R}^d}\eta\otimes\eta\,\tilde{f}(t,y,\mathrm{d}\eta)$. Therefore, we get in a similar way:

$$\mathcal{W}_1\left(\pi^{-\frac{d}{2}}\,\tilde{\rho}(t),\pi^{-\frac{d}{2}}\,\gamma^2\right)\leq\frac{C_0}{\sqrt{\ln t}}\qquad\forall t\geq 2.$$

— Introducing

$$I_1(t)=\int_{\mathbb{R}^d}\tilde{J}(t,\mathrm{d}y),\qquad I_2(t)=\int_{\mathbb{R}^d}y\,\tilde{\rho}(t,y)\,\mathrm{d}y,\qquad \tilde{I}_2=\tau\,I_2,$$

computations similar to those in the proof of Theorem 1.1.5 yield

$$\dot{I}_1=-2\lambda I_2,\qquad \dot{I}_2=\frac{1}{\tau^2(t)}I_1,\qquad \ddot{\tilde{I}}_2=0,\qquad I_2(t)=\frac{1}{\tau(t)}\left(I_1(0)\,t+I_2(0)\right)\underset{t\to\infty}{\longrightarrow}0=\int y\,\gamma^2(y)\,\mathrm{d}y.$$

Moreover, as soon as $I_1(0)\neq 0$, there holds

$$I_2(t)\underset{t\to\infty}{\sim}\frac{I_1(0)}{2\sqrt{\lambda\ln t}}.$$

In the same way, from the conservation of the energy for f by translating it into estimates on $\tilde{f}$, we derive for all $t\geq 2$:

$$\left|\int_{\mathbb{R}^d}|y|^2\,\tilde{\rho}(t,y)\,\mathrm{d}y-\int_{\mathbb{R}^d}|y|^2\,\gamma^2(y)\,\mathrm{d}y\right|\leq\frac{C_0}{\sqrt{\ln t}}.$$

It is interesting to see that the Wigner Measure found in Theorem 1.1.10 satisfy most of those properties. The only thing we could not prove is the convergence of the second momentum of the density, pointed out in Remark 1.1.12. If a good framework were found for (1.1.16) and if we could show the fact that the Wigner Measure satisfy (1.1.16) in this sense, we would (probably) be able to prove also the convergence of this momentum.

Remark 1.5.1. $\nabla_x(\ln\rho(t))$ is actually weakly defined $\rho(t)$-a.e.: indeed, for every $\phi\in W^{1,\infty}(\mathbb{R}^d)$,

$$\int\nabla_x(\ln\rho)(t,.)\,\phi\,d\rho(t)=-\int\rho(t,x)\,\nabla\phi(x)\,dx=-\int\nabla\phi\,d\rho(t).$$

In the same way, the term $\nabla_x(\ln\rho)\cdot\nabla_\xi f$ is weakly well-defined as soon as $\rho(t)\in W^{1,1}$ because for every $\phi\in L^\infty(\mathbb{R}^d_x,W^{1,\infty}(\mathbb{R}^d_\xi))$,

$$\begin{aligned}\langle\nabla_x(\ln\rho)(t,x)\cdot\nabla_\xi f(t,x,\xi),\phi(x,\xi)\rangle_{(x,\xi)}&=\langle\nabla_x(\ln\rho)(t,x)\,f(t,x,\xi),\nabla_\xi\phi(x,\xi)\rangle_{(x,\xi)}\\&=\left\langle\nabla_x(\ln\rho)(t,x),\langle f(t,x,\xi),\nabla_\xi\phi(x,\xi)\rangle_\xi\right\rangle_x,\end{aligned}$$

with the last term well-defined because:

$$\begin{aligned}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\nabla_x(\ln\rho)(t,x)\,f(t,x,\xi)\cdot\nabla_\xi\phi(x,\xi)|\,dxd\xi&\leq\int_{\mathbb{R}^d_x}|\nabla_x(\ln\rho)(t,x)|\int_{\mathbb{R}^d_\xi}|f(t,x,\xi)\,\nabla_\xi\phi(x,\xi)|\,d\xi\,dx\\&\leq\int_{\mathbb{R}^d_x}|\nabla_x(\ln\rho)(t,x)|\int_{\mathbb{R}^d_\xi}f(t,x,\xi)\,\|\nabla_\xi\phi\|_{L^\infty}\,d\xi\,dx\\&\leq\int_{\mathbb{R}^d_x}|\nabla_x(\ln\rho)(t,x)|\,\rho(t,x)\,\|\nabla_\xi\phi\|_{L^\infty}\,dx\\&\leq\int_{\mathbb{R}^d_x}|\nabla_x\rho(t,x)|\,\|\nabla_\xi\phi\|_{L^\infty}\,dx<\infty.\end{aligned}$$

Such remarks might help in order to find a real formalization of the equation, but this is not our goal here. However, we could not prove any $W^{1,1}$ regularity for ρ, whether for the Wigner Measure or with an estimate in the previous discussion.

1.5.2. Explicit solutions

Actually, there exists a particular case in which the Wigner Measure can be computed explicitly and is a solution to (1.1.16): the Gaussian case, providing *Gaussian-monokinetic* solutions to (1.1.16). It happens when all the initial data for (1.1.1) are Gaussian up to a quadratic complex oscillation. This result was proved by R. Carles and A. Nouri:

Theorem 1.5.2 ([36, Theorem 1.1.] and its proof)**.** *Let* λ, ρ_*, $\sigma_0 > 0$ *and* ω_0, $p_0 \in \mathbb{R}$. *Set*

$$\rho_{\text{in}}(x) = \rho_* e^{-\sigma_0 x^2}, \qquad \phi_{\text{in}} = \omega_0 \frac{x^2}{2} + p_0 x, \qquad v_{in}(x) = \phi_{\text{in}}'(x),$$

and consider the solution $\tau_0 \in \mathcal{C}^\infty(\mathbb{R}^+)$ *to the ordinary differential equation*

$$\ddot{\tau}_0 = \frac{2\lambda\sigma_0}{\tau_0}, \qquad \tau_0(0) = 1, \qquad \dot{\tau}_0(0) = \omega_0.$$

Set

$$\rho(t,x) = \frac{\rho_*}{\tau_0(t)} e^{-\sigma_0 \frac{(x-p_0 t)^2}{\tau_0(t)^2}}, \qquad v(t,x) = \frac{\dot{\tau}_0(t)}{\tau_0(t)}(x - p_0 t) + p_0$$

and consider u_ε *the solution to* (1.1.1) *with initial data*

$$u_{\varepsilon,\text{in}}(x) = \sqrt{\rho_{\text{in}}(x)}\, e^{i\,\frac{\phi_{\text{in}}(x)}{\varepsilon}} \in \mathcal{F}(H^1) \cap H^1(\mathbb{R}),$$

provided by Theorem 1.1.4. Then the Wigner Transform $W_\varepsilon(t)$ *of* $(u_\varepsilon(t))_{\varepsilon>0}$ *weakly converges (in terms of measures) when* $\varepsilon \to 0$ *for all* $t \geq 0$ *to the finite measure*

$$W(t, \mathrm{d}x, \mathrm{d}\xi) = \rho(t,x)\,\mathrm{d}x \otimes \delta_{\xi = v(t,x)},$$

solution to (1.1.16) *with* $W(0, \mathrm{d}x, \mathrm{d}\xi) = \rho_{\text{in}}(x)\,\mathrm{d}x \otimes \delta_{\xi = v_{in}(x)}$ *because* (ρ, v) *is solution to* (1.1.17).

The proof relies on the fact that the solution u_ε to (1.1.1) can actually be computed explicitly in this case, the Wigner Measure then readily follows from some computations. It is interesting to see that the initial data is a WKB state, satisfying (A2), and for this case this feature still holds for all time, recovering a (time-dependent) monokinetic measure for the Wigner Measure. Moreover, another interesting feature is the fact that the density (either for u_ε or for the Wigner Measure W) never vanishes, and even more: $\nabla(\log \rho_\varepsilon)$ is actually well defined for $\varepsilon \geq 0$ as an affine function in x, which is why we can say that this measure is solution to (1.1.16). Such a feature is very exceptional and cannot be extended to the general case, in particular for (1.1.1). However, we extend this class of solutions to (1.1.16) with a new class of explicit solutions, which are Gaussian in x multiplied by an x-dependent Gaussian in ξ for all time, stated in Theorem 1.1.15. We call them *Gaussian-Gaussian* solutions, by opposition with the previous Gaussian-monokinetic solutions.

1.5.3. Proof of Theorem 1.1.15

The main step of this proof is to prove the part 2 of Theorem 1.1.15. Indeed, the computations that will be done can be done reversely, or in another way one can prove directly by some easy computations that (1.1.20) is a solution to (1.1.16). We must also prove that c_1 solution to (1.1.18) is $C^\infty(\mathbb{R}^+)$, but this has already been done in [33].

With the notations and assumptions of the part 2 of Theorem 1.1.15, we compute:

$$\begin{aligned}
\partial_t f(t,x,\xi) = \Bigg[&-\frac{\dot{c}_1(t)}{c_1(t)} - \frac{\partial_t c_2(t,x)}{c_2(t,x)} + 2\,\frac{\dot{c}_1(t)\,(x - b_1(t))^2}{c_1(t)^3} + 2\,\frac{\dot{b}_1(t)\cdot(x-b_1(t))}{c_1(t)^2} \\
&+ 2\,\frac{\partial_t c_2(t,x)\,(\xi - b_2(t,x))^2}{c_2(t,x)^3} + 2\,\frac{\partial_t b_2(t,x)\cdot(\xi - b_2(t,x))}{c_2(t,x)^2} \Bigg] f(t,x,\xi), \\
\partial_x f(t,x,\xi) = \Bigg[&-2\,\frac{x - b_1(t)}{c_1(t)^2} + 2\,\partial_x b_2(t,x)\frac{\xi - b_2(t,x)}{c_2(t,x)^2} + 2\,\partial_x c_2(t,x)\frac{(\xi - b_2(t,x))^2}{c_2(t,x)^3} \Bigg] f(t,x,\xi), \\
\partial_\xi f(t,x,\xi) = &-2\,\frac{\xi - b_2(t,x)}{c_2(t,x)^2} f(t,x,\xi).
\end{aligned}$$

We also obviously get $\rho(t,x) = \frac{1}{\sqrt{\pi}\,c_1(t)} e^{-\frac{(x-b_1(t))^2}{c_1(t)^2}}$, therefore it is easy to compute:

$$\partial_x(\ln\rho)(t,x) = -2\,\frac{x - b_1(t)}{c_1(t)}.$$

Plugging all those identities into (1.1.16) leads to

$$0 = \left[-\frac{\dot c_1(t)}{c_1(t)} - \frac{\partial_t c_2(t,x)}{c_2(t,x)} + 2\,\frac{\dot c_1(t)\,(x-b_1(t))^2}{c_1(t)^3} + 2\,\frac{\dot b_1(t)\cdot(x-b_1(t))}{c_1(t)^2} + 2\,\frac{\partial_t c_2(t,x)\,(\xi - b_2(t,x))^2}{c_2(t,x)^3} \right.$$
$$+ 2\,\frac{\partial_t b_2(t,x)\cdot(\xi-b_2(t,x))}{c_2(t,x)^2} - 2\,\xi\,\frac{x-b_1(t)}{c_1(t)^2} + 2\,\partial_x b_2(t,x)\frac{(\xi-b_2(t,x))\,\xi}{c_2(t,x)^2} + 2\,\partial_x c_2(t,x)\frac{\xi\,(\xi-b_2(t,x))^2}{c_2(t,x)^3}$$
$$\left. -4\lambda\,\frac{\xi-b_2(t,x)}{c_2(t,x)^2}\,\frac{x-b_1(t)}{c_1(t)} \right] f(t,x,\xi),$$

which is of the form

$$P(t,x,\xi)f(t,x,\xi) = 0.$$

where P is a function such that for every (t,x), $P(t,x,.)$ is polynomial of degree at most 3. Since $f(t,x,\xi) > 0$ for every (t,x,ξ), there holds $P = 0$ and therefore for every (t,x), the coefficients of the polynomial function $P(t,x,.)$ are zero. In particular, the coefficient of highest degree is $2\,\partial_x c_2(t,x)\,c_2(t,x)^{-3}$, which yields

$$\partial_x c_2(t,x) = 0, \qquad \text{for all } t\in[0,T), x\in\mathbb{R},$$

and thus c_2 does not depend on x. We now take a more suitable basis to get zero coefficients for the polynomial function $\xi \mapsto P(t,x,\xi)$ of degree at most 2: $\big((\xi-b_2(t,x))^2, \xi-b_2(t,x), 1\big)$. Again, the coefficients in this basis are all zero, which yields for $(\xi-b_2(t,x))^2$:

$$2\frac{\dot c_2(t)}{c_2(t)^3} + 2\frac{\partial_x b_2(t,x)}{c_2(t,x)^2} = 0.$$

This equation leads to

$$\partial_x b_2(t,x) = -\frac{\dot c_2(t)}{c_2(t)},$$

and then, there exists a function $p_0 = p_0(t)$ such that:

$$b_2(t,x) = -\frac{\dot c_2(t)}{c_2(t)}\,x + p_0(t), \qquad \text{for all } t\in[0,T), x\in\mathbb{R}. \tag{1.5.4}$$

The assumption on the regularity of b_2 shows that $p_0 \in \mathcal{C}^1([0,T))$. But then, we also get:

$$\dot c_2(t) = c_2(t)\,(p_0(t) - b_2(t,1)) \in \mathcal{C}^1([0,T)).$$

Therefore, $c_2 \in \mathcal{C}^2([0,T))$. Now, examining the coefficient for $(\xi - b_2(t,x))$, we get

$$2\,\frac{\partial_t b_2(t,x)}{c_2(t)^2} - 2\,\frac{x-b_1(t)}{c_1(t)^2} + 2\,\frac{b_2(t,x)\,\partial_x b_2(t,x)}{c_2(t)^2} - 4\lambda\,\frac{x-b_1(t)}{c_1(t)^2\,c_2(t)^2} = 0, \qquad \text{for all } (t,x).$$

In terms of $\partial_t b_2$, this reads

$$\begin{aligned}\partial_t b_2(t,x) &= \left(1+\frac{2\lambda}{c_2(t)^2}\right)\frac{c_2(t)^2}{c_1(t)^2}\,(x-b_1(t)) - b_2(t,x)\,\partial_x b_2(t,x)\\ &= \left[\left(1+\frac{2\lambda}{c_2(t)^2}\right)\frac{c_2(t)^2}{c_1(t)^2} - \frac{\dot c_2(t)^2}{c_2(t)^2}\right] x - \left(1+\frac{2\lambda}{c_2(t)^2}\right)\frac{c_2(t)^2}{c_1(t)^2}\,b_1(t) + \frac{\dot c_2(t)}{c_2(t)}p_0(t).\end{aligned}$$

However, differentiating (1.5.4) with respect to t gives:

$$\partial_t b_2(t,x) = \left(-\frac{\ddot c_2(t)}{c_2(t)} + \frac{\dot c_2(t)^2}{c_2(t)^2}\right) x + \dot p_0(t).$$

This yields the following system of equations for all $t \geq 0$:

$$\begin{aligned}\left(1+\frac{2\lambda}{c_2(t)^2}\right)\frac{c_2(t)^2}{c_1(t)^2} &= -\frac{\ddot c_2(t)}{c_2(t)} + 2\,\frac{\dot c_2(t)^2}{c_2(t)^2},\\ \dot p_0(t) &= -\left(1+\frac{2\lambda}{c_2(t)^2}\right)\frac{c_2(t)^2}{c_1(t)^2}\,b_1(t) + \frac{\dot c_2(t)}{c_2(t)}p_0(t).\end{aligned} \tag{1.5.5}$$

In particular, the second equation shows that $\dot p_0 \in \mathcal{C}^1([0,T))$ (since the right-hand side is) and is actually an ordinary differential equation of order 1. The solution is well-known as soon as we remark that $\frac{\dot c_2}{c_2} = \frac{\mathrm{d}}{\mathrm{d}t}(\ln c_2)$ and reads:

$$p_0(t) = c_2(t)\left(C_0 - \int_0^t \left(1+\frac{2\lambda}{c_2(s)^2}\right)\frac{c_2(s)^2}{c_1(s)^2}\,\frac{b_1(s)}{c_2(s)}\,\mathrm{d}s\right)$$

and thanks to (1.5.5), we can expand it:

$$\begin{aligned}
p_0(t) &= c_2(t) \left(C_0 - \int_0^t \left(-\frac{\ddot{c}_2(s)}{c_2(s)} + 2\,\frac{\dot{c}_2(s)^2}{c_2(s)^2} \right) \frac{b_1(s)}{c_2(s)}\,\mathrm{d}s \right) \\
&= c_2(t) \left(C_0 + \int_0^t \frac{\mathrm{d}^2}{\mathrm{d}s^2} \left(\frac{1}{c_2(s)} \right) b_1(s)\,\mathrm{d}s \right) \\
&= c_2(t)\, C_1 + \frac{\dot{c}_2(t)}{c_2(t)}\, b_1(t) - c_2(t) \int_0^t \frac{\dot{c}_2(s)}{c_2(s)^2}\, \dot{b}_1(s)\,\mathrm{d}s,
\end{aligned}$$

where $C_1 = C_0 - \frac{\dot{c}_2(0)}{c_2(0)^2}\, b_1(0)$ with an integration by parts. Last, the constant in ξ gives the following equation:

$$-\frac{\dot{c}_1(t)}{c_1(t)} - \frac{\dot{c}_2(t)}{c_2(t)} + 2\,\frac{\dot{c}_1(t)}{c_1(t)^3}\,(x - b_1(t))^2 + 2\frac{\dot{b}_1(t)}{c_1(t)^2}\,(x - b_1(t)) - 2\,\frac{b_2(t,x)}{c_1(t)^2}(x - b_1(t)) = 0.$$

But since we know that b_2 is affine in x, the left-hand side is a polynomial function in x of degree 2 for all $t \in [0,T)$. Therefore, the coefficients in every basis are null. This time, we take the basis: $\big((x - b_1(t))^2, x - b_1(t), 1\big)$. For the first one and for the constants, we get

$$2\,\frac{\dot{c}_1(t)}{c_1(t)^3} + 2\,\frac{\dot{c}_2(t)}{c_2(t)\,c_1(t)^2} = 0, \qquad \text{and} \qquad -\frac{\dot{c}_1(t)}{c_1(t)} - \frac{\dot{c}_2(t)}{c_2(t)} = 0.$$

Those 2 equations actually reduce in a single one, which is

$$\frac{\mathrm{d}}{\mathrm{d}t}(c_1\, c_2) = 0,$$

and therefore, for all $t \in [0,T)$,

$$c_1(t)\, c_2(t) = c_1(0)\, c_2(0) =: \tilde{C} > 0.$$

We already know that c_2 is $\mathcal{C}^2$ and positive, therefore so is c_1. Coming back to (1.5.5), we now have

$$\ddot{c}_2 = 2\,\frac{\dot{c}_2^2}{c_2} - \frac{2\lambda}{\tilde{C}}\,c_2^3 - \frac{c_2^5}{\tilde{C}},$$

which reads in terms of c_1

$$\ddot{c}_1 = \frac{2\lambda}{c_1} + \frac{\tilde{C}^2}{c_1^3},$$

which is (1.1.18). Last, the final equation we have comes from the coefficient for $(x - b_1(t))$:

$$2\,\frac{\dot{b}_1(t)}{c_1(t)^2} + 2\,\frac{\dot{c}_2(t)}{c_1(t)^2\, c_2(t)}\, b_1(t) - 2\frac{p_0(t)}{c_1(t)^2} = 0, \qquad \text{for all } t \geq 0.$$

This leads to

$$\dot{b}_1 = -\frac{\dot{c}_2}{c_2}\, b_1 + p_0.$$

All the terms in the right-hand side are $\mathcal{C}^1([0,T))$, therefore so is $\dot{b}_1$, which yields to the $\mathcal{C}^2$-regularity of b_1. Hence, we can again expand the expression for p_0 found previously with another integration by parts:

$$p_0(t) = C_2\, c_2(t) + \frac{\dot{c}_2(t)}{c_2(t)}\, b_1(t) + \dot{b}_1(t) - c_2(t) \int_0^t \frac{\ddot{b}_1(s)}{c_2(s)}\, ds,$$

with $C_2 = C_1 + \frac{1}{c_2(0)}$ Plugging this expression of p_0 into the expression of $\dot{b}_1$ leads to

$$C_2\, c_2(t) = c_2(t) \int_0^t \frac{\ddot{b}_1(s)}{c_2(s)}\, ds.$$

Since $c_2 > 0$, we then obtain $C_2 = 0$ and $\frac{\ddot{b}_1}{c_2} = 0$, which is $\ddot{b}_1 = 0$. Thus, there exists B_0, B_1 constants such that

$$b_1 = B_1\, t + B_0,$$

and this gives the final expression for p_0 (and therefore for b_2):

$$p_0(t) = (B_1\, t + B_0)\frac{\dot{c}_2(t)}{c_2(t)} + B_1.$$

Putting all together leads to (1.1.18)-(1.1.19), which yields the $\mathcal{C}^\infty$ feature of all the functions.

The last thing we need to check the convergence rate of $\tilde{\rho}(t)$ to γ^2 in L^1. For this, we can use again the Csiszár-Kullback inequality, and compute with the expression of $\tilde{\rho} = \tau(t)\,\rho(t, \tau(t)y)$:

$$\begin{aligned}
\left\|\tilde{\rho}^2(t) - \gamma^2\right\|_{L^1}^2 &\leq 2\|\gamma^2\|_{L^1} \int_{\mathbb{R}} \gamma^2(y) \ln \frac{\gamma^2(y)}{\tilde{\rho}(t,y)}\, \mathrm{d}y \\
&\leq 2\sqrt{\pi} \int_{\mathbb{R}} \left[\frac{(\tau(t)y - b_1(t))^2}{c_1(t)^2} - y^2 + \ln \frac{c_1(t)}{\tau(t)}\right] e^{-y^2}\, \mathrm{d}y \\
&\leq 2\pi \left[\frac{1}{2}\left(1 - \left(\frac{\tau(t)}{c_1(t)}\right)^2\right) + \ln \frac{c_1(t)}{\tau(t)} + \frac{b_1(t)^2}{c_1(t)^2}\right].
\end{aligned}$$

From [33], it is known that both $\tau(t)$ and $c_1(t)$ have the same feature when $t \to \infty$:

$$\tau(t) = 2t\sqrt{\lambda \ln t}\left(1 + \mathrm{O}\left(\frac{\ln \ln t}{\ln t}\right)\right) = c_1(t).$$

Therefore, we get

$$1 - \left(\frac{\tau(t)}{c_1(t)}\right)^2 = \mathrm{O}\left(\frac{\ln \ln t}{\ln t}\right), \qquad \ln \frac{c_1(t)}{\tau(t)} = \mathrm{O}\left(\frac{\ln \ln t}{\ln t}\right).$$

Moreover, since $b_1 = B_1\, t + B_0$, it is known that

$$\frac{b_1(t)^2}{c_1(t)^2} = \mathrm{O}\left(\frac{1}{\ln t}\right).$$

Putting everything together, we get (1.1.21). □

Appendix

1.A. Proof of Proposition 1.2.2

We now prove the points 1 to 3 of Proposition 1.2.2. The part 1 is proven in Section 1.A.1, Section 1.A.2 is devoted to the proof of part 2, and finally we prove part 3 in Section 1.A.3.

1.A.1. First part: proof of the second momentum in ξ

The proof of part 1 of Proposition 1.2.2 is organized in 4 parts. First, we will prove the equality of $\int_{\mathbb{R}^d} |\xi|^2 \, W_\varepsilon^H(x,\xi)\,\mathrm{d}\xi$ for $f_\varepsilon \in \mathcal{S}(\mathbb{R}^d)$ because we need better regularity for the interchange of integrals we will do. Then, we will generalize this result to the case $f_\varepsilon \in H^1$ by using an argument of continuity of a quadratic form and the fact that the integral is still well-defined even if $f_\varepsilon \in H^1$ because $|\xi|^2 \, W_\varepsilon^H(x,\xi) \geq 0$. Then we will be able to consider $\xi_i \xi_j \, W_\varepsilon^H(x,\xi)$ without any issue, and we will prove the equality involving it in the same way: first for $f_\varepsilon \in \mathcal{S}(\mathbb{R}^d)$, and then generalizing it for $f_\varepsilon \in H^1$ thanks to a continuity argument.

Scalar second momentum: $\mathcal{S}(\mathbb{R}^d)$ case. As W_ε^H is non-negative, we can consider $\int_{\mathbb{R}^d} |\xi|^2 \, W_\varepsilon^H(x,\xi)\,\mathrm{d}\xi$ without any issue. Moreover, we suppose here that $f_\varepsilon \in \mathcal{S}(\mathbb{R}^d)$. Then:

$$\int_{\mathbb{R}^d} |\xi|^2 \, W_\varepsilon^H(x,\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^d} |\xi|^2 \, (W_\varepsilon *_\xi \gamma_\varepsilon *_x \gamma_\varepsilon)(x,\xi)\,\mathrm{d}\xi = \left(\int_{\mathbb{R}^d} |\xi|^2 \, W_\varepsilon *_\xi \gamma_\varepsilon \,\mathrm{d}\xi\right) *_x \gamma_\varepsilon(x).$$

We check that the previous integral exchange is rigorous.

$$\begin{aligned}
&\left(\int_{\mathbb{R}^d} |\xi|^2 \; |W_\varepsilon *_\xi \gamma_\varepsilon| \,\mathrm{d}\xi\right) * \gamma_\varepsilon(x) \\
&\qquad = \left(\int_{\mathbb{R}^d} |\xi|^2 \left| \mathcal{F}_{z\to\xi}\left(f_\varepsilon\left(.+\frac{\varepsilon}{2}\,z\right) \overline{f_\varepsilon\left(.-\frac{\varepsilon}{2}\,z\right)}\right) *_\xi \mathcal{F}_{z\to\xi}\left(\exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right)\right| \mathrm{d}\xi\right) * \gamma_\varepsilon(x) \\
&\qquad = \left(\int_{\mathbb{R}^d} \left| \mathcal{F}_{z\to\xi}\left(\Delta_z\left(f_\varepsilon\left(.+\frac{\varepsilon}{2}\,z\right) \overline{f_\varepsilon\left(.-\frac{\varepsilon}{2}\,z\right)} \exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right)\right)\right| \mathrm{d}\xi\right) * \gamma_\varepsilon(x) \\
&\qquad \leq C_0 \left\| \Delta_z\left(f_\varepsilon\left(.+\frac{\varepsilon}{2}\,z\right) \overline{f_\varepsilon\left(.-\frac{\varepsilon}{2}\,z\right)} \exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right)\right\|_{W_z^{d+1,1}} *_x \gamma_\varepsilon(x) \\
&\qquad \leq C_0 \left\| f_\varepsilon\left(.+\frac{\varepsilon}{2}\,z\right) \overline{f_\varepsilon\left(.-\frac{\varepsilon}{2}\,z\right)} \exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right\|_{W_z^{d+3,1}} *_x \gamma_\varepsilon(x) \\
&\qquad \leq C_0 \left\| \exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right\|_{W_z^{d+3,\infty}} \left\| f_\varepsilon\left(.+\frac{\varepsilon}{2}\,z\right) \overline{f_\varepsilon\left(.-\frac{\varepsilon}{2}\,z\right)}\right\|_{W_z^{d+3,1}} *_x \gamma_\varepsilon(x) \\
&\qquad \leq C_0 \left\| \exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right\|_{W_z^{d+1,\infty}} \left\| f_\varepsilon\left(.+\frac{\varepsilon}{2}\,z\right)\right\|_{H_z^{d+3}} \left\| \overline{f_\varepsilon\left(.-\frac{\varepsilon}{2}\,z\right)}\right\|_{H_z^{d+3}} *_x \gamma_\varepsilon(x) \\
&\qquad \leq C_0 \, \varepsilon^{-d} \left\| \exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right\|_{W_z^{d+1,\infty}} \|f_\varepsilon\|_{H^{d+3}}^2 \; \|\gamma_\varepsilon\|_{L^1} < \infty.
\end{aligned}$$

Remark 1.A.1. This computation shows that we actually only need $f_\varepsilon \in H^{d+3}$.

Now, come back to our first identity. We can compute in the way we want:

$$\int_{\mathbb{R}^d} |\xi|^2 \, W_\varepsilon^H(x,\xi)\,\mathrm{d}\xi = \left(\int_{\mathbb{R}^d} |\xi|^2 \, W_\varepsilon *_\xi \gamma_\varepsilon \,\mathrm{d}\xi\right) *_x \gamma_\varepsilon(x)$$

$$= -\left(\int_{\mathbb{R}^d} \mathcal{F}_{z\to\xi}\left(\Delta_z\left(f_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{f_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}\exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right)\right)\mathrm{d}\xi\right) *_x \gamma_\varepsilon(x)$$
$$= -\left[\Delta_z\left(f_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{f_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}\exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right)\right]\Bigg|_{z=0} *_x \gamma_\varepsilon(x).$$

Computing $\Delta_z\left(f_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)\overline{f_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)}\exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right)$, we obtain

$$\Delta_z\left(f_\varepsilon\left(x+\frac{\varepsilon}{2}z\right)\overline{f_\varepsilon\left(x-\frac{\varepsilon}{2}z\right)}\exp\left(-\varepsilon\frac{|z|^2}{4}\right)\right)\Bigg|_{z=0}$$
$$= \frac{\varepsilon^2}{4}\left[\Delta f_\varepsilon(x)\,\overline{f_\varepsilon(x)} + f_\varepsilon(x)\,\overline{\Delta f_\varepsilon(x)} - 2\left|\nabla f_\varepsilon(x)\right|^2\right] - \frac{\varepsilon d}{2}|f_\varepsilon(x)|^2$$
$$= \frac{\varepsilon^2}{4}\left[\Delta\left(|f_\varepsilon|^2\right)(x) - 4\left|\nabla f_\varepsilon(x)\right|^2\right] - \frac{\varepsilon d}{2}|f_\varepsilon(x)|^2.$$

Therefore, knowing that $\gamma_\varepsilon \in \mathcal{S}(\mathbb{R}^d)$, we can pass the Δ to the other side of the convolution and get (1.2.3). Keeping in mind that $\gamma_\varepsilon \in \mathcal{S}$, integrating in x yields (1.2.4).

Scalar second momentum: H^1 case. In the same way, we can still consider $\int_{\mathbb{R}^d} |\xi|^2\, W_\varepsilon^H(x,\xi)\,\mathrm{d}\xi$ even for $f_\varepsilon \in H^1$. However, it could still be equal to $+\infty$. The first part will be to show that this is not the case.

For fixed $\varepsilon > 0$, take a sequence of functions $f_{\varepsilon,k}$ in $\mathcal{S}(\mathbb{R}^d)$ converging to f_ε in H^1 when $k \to \infty$. Using the notation $W_{\varepsilon,k}$ (resp. $W_{\varepsilon,k}^H$) for the Wigner Transform (resp. the Husimi Transform) of the functions of the sequence, we first show that they converge uniformly to the Wigner Transform W_ε (resp. the Husimi Transform W_ε^H) of f_ε.

For any $x, \xi \in \mathbb{R}^d$

$$\begin{aligned}
|W_{\varepsilon,k}(x,\xi) - W_\varepsilon(x,\xi)| &\le C_0\left\| f_{\varepsilon,k}(x+\frac{\varepsilon z}{2})\,\overline{f_{\varepsilon,k}(x-\frac{\varepsilon z}{2})} - f_\varepsilon(x+\frac{\varepsilon z}{2})\,\overline{f_\varepsilon(x-\frac{\varepsilon z}{2})}\right\|_{L^1_z}\\
&\le C_0\left(\left\| f_{\varepsilon,k}(x+\frac{\varepsilon z}{2})\,\left(\overline{f_{\varepsilon,k}(x-\frac{\varepsilon z}{2}) - f_\varepsilon(x-\frac{\varepsilon z}{2})}\right)\right\|_{L^1_z}\right.\\
&\qquad\left. + \left\|\left(f_{\varepsilon,k}(x+\frac{\varepsilon z}{2}) - f_\varepsilon(x+\frac{\varepsilon z}{2})\right)\overline{f_\varepsilon(x-\frac{\varepsilon z}{2})}\right\|_{L^1_z}\right)\\
&\le C_0\left(\left\| f_{\varepsilon,k}(x+\frac{\varepsilon z}{2})\right\|_{L^2_z}\left\| f_{\varepsilon,k}(x-\frac{\varepsilon z}{2}) - f_\varepsilon(x-\frac{\varepsilon z}{2})\right\|_{L^2_z}\right.\\
&\qquad\left. + \left\| f_{\varepsilon,k}(x+\frac{\varepsilon z}{2}) - f_\varepsilon(x+\frac{\varepsilon z}{2})\right\|_{L^2_z}\left\| f_\varepsilon(x-\frac{\varepsilon z}{2})\right\|_{L^2_z}\right)\\
&\le C_\varepsilon\left(\|f_{\varepsilon,k}\|_{L^2}\|f_{\varepsilon,k} - f_\varepsilon\|_{L^2} + \|f_{\varepsilon,k} - f_\varepsilon\|_{L^2}\|f_\varepsilon\|_{L^2}\right)\\
&\le C_\varepsilon\|f_{\varepsilon,k} - f_\varepsilon\|_{L^2}.
\end{aligned}$$

Therefore, $W_{\varepsilon,k}$ converges uniformly to W_ε with the estimate

$$\|W_{\varepsilon,k} - W_\varepsilon\|_{L^\infty} \le C_\varepsilon\|f_{\varepsilon,k} - f_\varepsilon\|_{L^2},$$

and the same kind of estimate holds for the Husimi Transform:

$$\|W_{\varepsilon,k}^H - W_\varepsilon^H\|_{L^\infty} = \|(W_{\varepsilon,k} - W_\varepsilon) * G_\varepsilon\|_{L^\infty} \le \|W_{\varepsilon,k} - W_\varepsilon\|_{L^\infty}\|G_\varepsilon\|_{L^1} \le C_\varepsilon\|f_{\varepsilon,k} - f_\varepsilon\|_{L^2}.$$

Hence, Fatou's lemma for $|\xi|^2 W_{\varepsilon,k}^H(x,\xi)$ yields

$$\int_{\mathbb{R}^d} |\xi|^2\, W_\varepsilon^H(x,\xi)\,\mathrm{d}\xi \le \liminf_{k\to\infty}\int_{\mathbb{R}^d} |\xi|^2\, W_{\varepsilon,k}^H(x,\xi)\,\mathrm{d}\xi.$$

The previous computation yields

$$\int_{\mathbb{R}^d} |\xi|^2\, W_{\varepsilon,k}^H(x,\xi)\,\mathrm{d}\xi = \varepsilon^2\,|\nabla f_{\varepsilon,k}|^2 * \gamma_\varepsilon(x) - \frac{\varepsilon^2}{4}\,|f_{\varepsilon,k}|^2 * \Delta\gamma_\varepsilon(x) + \frac{\varepsilon d}{2}|f_{\varepsilon,k}|^2 * \gamma_\varepsilon(x).$$

But $f_{\varepsilon,k} \underset{k\to\infty}{\longrightarrow} f_\varepsilon$ in H^1, so $|\nabla f_{\varepsilon,k}|^2 \underset{k\to\infty}{\longrightarrow} |\nabla u_\varepsilon|^2$ and $|f_{\varepsilon,k}|^2 \underset{k\to\infty}{\longrightarrow} |f_\varepsilon|^2$ in L^1, therefore:

$$\int_{\mathbb{R}^d} |\xi|^2\, W_\varepsilon^H(x,\xi)\,\mathrm{d}\xi \le \varepsilon^2\,|\nabla f_\varepsilon|^2 * \gamma_\varepsilon(x) - \frac{\varepsilon^2}{4}\,|f_\varepsilon|^2 * \Delta\gamma_\varepsilon(x) + \frac{\varepsilon d}{2}|f_\varepsilon|^2 * \gamma_\varepsilon(x) < \infty.$$

Therefore, the map

$$\begin{aligned} H^1 &\to \mathbb{R}^+ \\ f_\varepsilon &\mapsto \int_{\mathbb{R}^d} |\xi|^2 \, W_\varepsilon^H(x,\xi) \, \mathrm{d}\xi \end{aligned}$$

is well-defined for every $x \in \mathbb{R}^d$. Moreover, it is a non-negative quadratic form because W_ε and then also W_ε^H are quadratic. Furthermore, it is continuous thanks to the previous inequality which leads to

$$\int_{\mathbb{R}^d} |\xi|^2 \, W_\varepsilon^H(x,\xi) \, \mathrm{d}\xi \le C_\varepsilon \|f_\varepsilon\|_{H^1}^2.$$

Thus, the equality (1.2.3), which is true in $\mathcal{S}(\mathbb{R}^d)$ dense subspace in H^1, also holds in H^1.

Vector second momentum: $\mathcal{S}$ case. With the same assumptions, we can consider $\int_{\mathbb{R}^d} \xi_i \xi_j \, W_\varepsilon^H(x,\xi) \, \mathrm{d}\xi$ as we now know that $\xi_i\xi_j \, W_\varepsilon^H(x,\xi)$ is integrable thanks to the previous identity, and in the same way, we have for $f_\varepsilon \in \mathcal{S}(\mathbb{R}^d)$ and for every $x \in \mathbb{R}^d$:

$$\begin{aligned} \int_{\mathbb{R}^d} \xi_i \xi_j \, W_\varepsilon^H(x,\xi) \, \mathrm{d}\xi &= \int_{\mathbb{R}^d} \xi_i \xi_j \, (W_\varepsilon *_\xi \gamma_\varepsilon *_x \gamma_\varepsilon)(x,\xi) \, \mathrm{d}\xi \\ &= \left(\int_{\mathbb{R}^d} \xi_i \xi_j \, W_\varepsilon *_\xi \gamma_\varepsilon \, \mathrm{d}\xi \right) *_x \gamma_\varepsilon(x), \end{aligned}$$

the interchange of integral is rigorous with the same kind of estimate as previously. Moreover, we readily compute

$$\begin{aligned} &\left[\partial_{z_i} \partial_{z_j} \left(f_\varepsilon\left(x + \frac{\varepsilon}{2}\, z\right) \overline{f_\varepsilon\left(x - \frac{\varepsilon}{2}\, z\right)} \exp\left(-\varepsilon \frac{|z|^2}{4}\right) \right) \right] \Big|_{z=0} \\ &\qquad = \frac{\varepsilon^2}{4} \left[\partial_i \partial_j f_\varepsilon\,(x) \, \overline{f_\varepsilon\,(x)} + f_\varepsilon\,(x) \, \overline{\partial_i \partial_j f_\varepsilon\,(x)} - \partial_i f_\varepsilon\,(x) \, \overline{\partial_j f_\varepsilon\,(x)} - \partial_j f_\varepsilon\,(x) \, \overline{\partial_i f_\varepsilon\,(x)} \right] - \frac{\varepsilon \delta_{ij}}{2} f_\varepsilon\,(x) \, \overline{f_\varepsilon\,(x)} \\ &\qquad = \frac{\varepsilon^2}{4} \left[\partial_i \partial_j \left(|f_\varepsilon|^2\right)(x) - 4 \,\mathrm{Re} \left(\partial_i f_\varepsilon\,(x) \, \overline{\partial_j f_\varepsilon\,(x)} \right) \right] - \frac{\varepsilon \delta_{ij}}{2} |f_\varepsilon(x)|^2. \end{aligned}$$

Therefore, in the same way as in the previous first section, we get (1.2.5) and (1.2.6).

Vector second momentum: H^1 case. The generalization of this equality is similar to the end of the previous generalization for the scalar second momentum. The map

$$\begin{aligned} H^1 &\to \mathbb{R} \\ f_\varepsilon &\mapsto \int_{\mathbb{R}^d} \xi_i \xi_j \, W_\varepsilon^H(x,\xi) \, \mathrm{d}\xi \end{aligned}$$

is a well-defined, continuous quadratic form thanks to the previous equality for the scalar second momentum. Then, the identities found for $f_\varepsilon \in \mathcal{S}(\mathbb{R}^d)$ also hold for $f_\varepsilon \in H^1$.

1.A.2. Second part: first momentum in ξ

We know that $\int_{\mathbb{R}^d} |\xi|^2 \, W_\varepsilon^H(x,\xi) \, \mathrm{d}\xi < \infty$ by the previous proof and also that $\int_{\mathbb{R}^d} W_\varepsilon^H(x,\xi) \, \mathrm{d}\xi < \infty$, therefore we can consider $\int_{\mathbb{R}^d} \xi \, W_\varepsilon^H(x,\xi) \, \mathrm{d}\xi$. Then:

$$\begin{aligned} \int_{\mathbb{R}^d} \xi \, W_\varepsilon^H(x,\xi) \, \mathrm{d}\xi &= \int_{\mathbb{R}^d} \xi \, (W_\varepsilon *_\xi \gamma_\varepsilon *_x \gamma_\varepsilon)(x,\xi) \, \mathrm{d}\xi \\ &= \left(\int_{\mathbb{R}^d} \xi \, W_\varepsilon *_\xi \gamma_\varepsilon \, \mathrm{d}\xi \right) *_x \gamma_\varepsilon, \end{aligned}$$

the integral exchange being rigorous for $f_\varepsilon \in \mathcal{S}(\mathbb{R}^d)$ with the same kind of computation as before, which infers that:

$$\begin{aligned} \int_{\mathbb{R}^d} \xi \, W_\varepsilon^H(x,\xi) \, \mathrm{d}\xi &= \left(-i \, \nabla_z \left(f_\varepsilon\left(\,.+ \frac{\varepsilon}{2}\, z\right) \overline{f_\varepsilon\left(\,.- \frac{\varepsilon}{2}\, z\right)} \exp\left(-\varepsilon \frac{|z|^2}{4}\right) \right) \Big|_{z=0} \right) *_x \gamma_\varepsilon \\ &= \varepsilon \,\mathrm{Im} \left(\nabla f_\varepsilon \, \overline{f_\varepsilon} \right) * \gamma_\varepsilon(x), \end{aligned}$$

and therefore (1.2.7) for $f_\varepsilon \in \mathcal{S}(\mathbb{R}^d)$, (1.2.8) being obvious by integrating this result. The conclusion for the general case runs as before.

1.A.3. Third part: second momentum in x

In the same way, since W_ε^H is non-negative, we have, thanks to Proposition 1.2.1,

$$\begin{aligned}\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x|^2\, W_\varepsilon^H(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi &= \int_{\mathbb{R}^d} |x|^2 \left(\int_{\mathbb{R}^d} W_\varepsilon^H(x,\xi)\,\mathrm{d}\xi\right)\mathrm{d}x \\ &= \int_{\mathbb{R}^d} |x|^2\, |f_\varepsilon|^2 * \gamma_\varepsilon(x)\,\mathrm{d}x \\ &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x|^2\, |f_\varepsilon(x-y)|^2 * \gamma_\varepsilon(y)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Therefore, we can easily compute

$$\begin{aligned}\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x|^2\, W_\varepsilon^H(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} (|x-y|^2 + 2\,(x-y)\cdot y + |y|^2)\, |f_\varepsilon(x-y)|^2 * \gamma_\varepsilon(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \left(\int_{\mathbb{R}^d} |x|^2\, |f_\varepsilon(x)|^2\,\mathrm{d}x\right) \|\gamma_\varepsilon\|_{L^1} + 2\left(\int_{\mathbb{R}^d} x\, |f_\varepsilon(x)|^2\,\mathrm{d}x\right)\cdot\left(\int_{\mathbb{R}^d} y\,\gamma_\varepsilon(y)\,\mathrm{d}y\right) \\ &\qquad + \|f_\varepsilon\|_{L^2}^2 \int_{\mathbb{R}^d} |y|^2\,\gamma_\varepsilon(y)\,\mathrm{d}y \\ &= \big\||x|^2|f_\varepsilon(x)|^2\big\|_{L^1} + \frac{\varepsilon d}{2}\,\|f_\varepsilon\|_{L^2}^2.\end{aligned}$$

thanks to the properties on the momenta of γ_ε.

Chapitre 2

The focusing logarithmic Schrödinger equation : analysis of breathers and nonlinear superposition

Abstract. We consider the logarithmic Schrödinger equation in the focusing regime. For this equation, Gaussian initial data remains Gaussian. In particular, the Gausson - a time-independent Gaussian function - is an orbitally stable solution. In the general case in dimension $d = 1$, the solution with Gaussian initial data is periodic, and we compute some approximations of the period in the case of small and large oscillations, showing that the period can be as large as wanted for the latter. The main result of this article is a principle of nonlinear superposition: starting from an initial data made of the sum of several standing Gaussian functions far from each other, the solution remains close (in L^2) to the sum of the corresponding Gaussian solutions for a long time, in square of the distance between the Gaussian functions.

2.1. Introduction

2.1.1. Setting

We are interested in the *Logarithmic Non-Linear Schrödinger Equation*

$$i\,\partial_t u + \frac{1}{2}\Delta u + \lambda u \ln |u|^2 = 0, \qquad u_{|t=0} = u_{\text{in}}, \tag{2.1.1}$$

with $x \in \mathbb{R}^d$, $d \geq 1$, $\lambda \in \mathbb{R} \setminus \{0\}$. It was introduced as a model of nonlinear wave mechanics and in nonlinear optics ([23], see also [25, 95, 98, 102, 59]). The case $\lambda < 0$ (whose study goes back to [40, 86]) was recently studied by R. Carles and I. Gallagher who made explicit an unusually faster dispersion with a universal behaviour of the modulus of the solution (see [33]). The knowledge of this behaviour was very recently improved with a convergence rate but also extended through the semiclassical limit in [67]. On the other hand, the case $\lambda > 0$ seems to be the more interesting from a physical point of view and has been studied formally and rigorously a lot (see for instance [58, 98]). In particular, the existence and uniqueness of solutions to the Cauchy problem have been solved in [40]. Moreover, it has been proved to be the focusing case and, in this context, a usual question is the existence of stationary states called *solitons* and their stability. For this equation, we know that the so called *Gausson*

$$G^d(x) := \exp\Big(\frac{d}{2} - \lambda |x|^2\Big), \qquad x \in \mathbb{R}^d, \tag{2.1.2}$$

and its by-products through the invariants of the equation (translation in space, Galilean invariance, multiplication by a complex constant of modulus 1) are explicit solutions to 2.1.1 and bound states for the energy functional. Several results address the orbital stability of the Gausson as well as the existence of other stationary solutions to 2.1.1; see e.g. [23, 38, 58, 8]. However, these solutions are not the only explicit solutions: indeed, any Gaussian initial data remains Gaussian and can be made explicit up to a matrix ODE. In this article, we study a particular class of time-dependent Gaussian solutions to 2.1.1 already introduced in [23] which are (almost) periodic in time in dimension 1, so called *breathers*, and we give a partial result for the nonlinear superposition of Gaussian solutions.

Remark 2.1.1 (Effect of scaling factors). As noticed in [33], unlike what happens in the case of an homogeneous nonlinearity (classically of the form $|u|^p u$), replacing u with κu ($\kappa > 0$) in 2.1.1 has only little effect, since we have

$$i\,\partial_t(\kappa u) + \frac{1}{2}\Delta(\kappa u) + \lambda(\kappa u) \ln |\kappa u|^2 - 2\lambda(\ln \kappa)\kappa u = 0.$$

The scaling factor thus corresponds to a purely time-dependent gauge transform:

$$\kappa u(t,x)e^{-2it\lambda\ln\kappa}$$

solves 2.1.1 (with initial datum κu_0). In particular, the L^2-norm of the initial datum does not influence the dynamics of the solution.

2.1.2. The Logarithmic Non-Linear Schrödinger Equation

The Logarithmic
Schrödinger Equation was introduced by I. Białynicki-Birula and J. Mycielski [23] who proved that it is the only nonlinear theory in which *the separability of noninteracting systems* hold: for noninteracting subsystems, no correlation is introduced by the nonlinear term. This means that for any initial data of the form $u_{\text{in}} = u^1_{\text{in}} \otimes u^2_{\text{in}}$, i.e. $u_{\text{in}}(x) = u^1_{\text{in}}(x_1)\, u^2_{\text{in}}(x_2)$ for all $x_1 \in \mathbb{R}^{d_1}$, $x_2 \in \mathbb{R}^{d_2}$ and $x = (x_1, x_2)$, the solution u to 2.1.1 (in dimension $d = d_1 + d_2$) is $u(t) = u^1(t) \otimes u^2(t)$ where u^1 and u^2 are the solutions to 2.1.1 in dimension d_1 (resp. d_2) with initial data u^1_{in} and u^2_{in} respectively.

They also emphasized that the case $\lambda > 0$ is probably the most physically relevant, and we will mathematically study this case in the rest of this paper. For this case, the Cauchy problem has already been studied in [40] (see also [39]). We define the energy space

$$W(\mathbb{R}^d) := \left\{ v \in H^1(\mathbb{R}^d), |v|^2 \ln |v|^2 \in L^1(\mathbb{R}^d) \right\}.$$

It is a reflexive Banach space when endowed with a Luxembourg type norm (see [38]). We can also define the mass and energy for all $v \in W(\mathbb{R}^d)$:

$$M(v) := \|v\|^2_{L^2}, \qquad E(v) := \frac{1}{2}\|\nabla v\|^2_{L^2} - \lambda \int_{\mathbb{R}^d} |v|^2(\ln|v|^2 - 1)\,\mathrm{d}x.$$

Theorem 2.1.2 ([40, Théorème 2.1], see also [39, Theorem 9.3.4])**.** *For any initial data $u_{\text{in}} \in W(\mathbb{R}^d)$, there exists a unique, global solution $u \in \mathcal{C}_b(\mathbb{R}, W(\mathbb{R}^d))$. Moreover the mass $M(u(t))$ and the energy $E(u(t))$ are independent of time.*

Note that whichever the sign of λ is, the energy E has no definite sign. The distinction between focusing or defocusing nonlinearity is thus a priori ambiguous. However, this ambiguity has already been removed by [38] (case $\lambda > 0$) and [33] (case $\lambda < 0$). Indeed, in the latter, the authors show that all the solutions disperse in an unusually faster way with a universal dynamic: after rescaling, the modulus of the solution converges to a universal Gaussian profile. On the other hand, it has been proved that there is no dispersion for large times for $\lambda > 0$ thanks to the following result, and hence it is the focusing case.

Lemma 2.1.3 ([38, Lemma 3.3])**.** *Let $\lambda > 0$. For any $k < \infty$ such that*

$$L_k := \left\{ v \in W(\mathbb{R}^d), \|v\|_{L^2} = 1, E(v) \le k \right\} \neq \emptyset,$$

there holds

$$\inf_{\substack{v \in L_k \\ 1 \le p \le \infty}} \|v\|_{L^p} > 0.$$

This lemma, along with the conservation of the energy and the invariance through scaling factors (with Remark 2.1.1), indicates that the solution to 2.1.1 is not dispersive, no matter how small the initial data are. For instance, its L^∞ norm is bounded from below: to be more precise, there holds for all $t \in \mathbb{R}$ (see the proof of the above result)

$$\|u(t)\|_{L^\infty} \ge \exp\Big[-\frac{E(u(t))}{2M(u(t))}\Big] = \exp\Big[-\frac{E(u_{\text{in}})}{2M(u_{\text{in}})}\Big].$$

Actually, a specific Gaussian function 2.1.2 called *Gausson* and its by-products through the invariants of the equation and the scaling effect, defined by

$$G^d_{\omega,x_0,v,\theta}(t,x) = \exp\left[i\left(\theta + 2\lambda\omega t - v\cdot x + \frac{|v|^2}{2}t\right) + \omega - \lambda|x - x_0 - vt|^2 \right],$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, and for any $\omega, \theta \in \mathbb{R}$, $x_0, v \in \mathbb{R}^d$, are known to be solutions to 2.1.1, as proved in [58] (and already noticed in [23]). It has also been proved that other radial stationary solutions to 2.1.1 exist in dimension $d \ge 3$ (see [58]), but the Gausson is clearly special since it is the unique positive $\mathcal{C}^2$ stationary solution to 2.1.1 (also proved in [58]) and also since it is orbitally stable ([8], following the work of [38]).

Theorem 2.1.4 ([8, Theorem 1.5])**.** *Let $\omega \in \mathbb{R}$. For any $\varepsilon > 0$, there exists $\eta > 0$ such that for all $u_0 \in W(\mathbb{R}^d)$ satisfying*

$$\inf_{\theta, x_0} \left\| u_0 - e^{\omega + i\theta} G^d(. - x_0) \right\|_{W(\mathbb{R}^d)} < \eta,$$

the solution $u(t)$ of 2.1.1 with initial data u_0 satisfies

$$\sup_t \inf_{\theta, x_0} \left\| u(t) - e^{\omega + i\theta} G^d(. - x_0) \right\|_{W(\mathbb{R}^d)} < \varepsilon.$$

2.1.3. Main results

From now on, we assume $\lambda > 0$.

Existence of breathers. The Gausson is an explicit important solution to 2.1.1. However, it is not the only solution that can be made explicit:

Proposition 2.1.5. *Any Gaussian initial data*

$$\exp\Big[\frac{d}{2} - x^\top A_{\text{in}} x\Big], \tag{2.1.3}$$

with $A_{\text{in}} \in S_d(\mathbb{C}) := \big\{M \in M_d(\mathbb{C}), M^\top = M\big\}$ *(where* $^\top$ *designates the transposition) such that* $\operatorname{Re} A_{\text{in}}$ *is positive definite, gives rise to a Gaussian solution to 2.1.1 of the form*

$$u^{A_{\text{in}}}(t,x) := b(t) \exp\Big[\frac{d}{2} - x^\top A(t) x\Big]. \tag{2.1.4}$$

Furthermore, $b(t)$ is explicitly given by the knowledge of A whereas the evolution of A is given by a first-order matrix ordinary differential equations (see the system of equations (6.14), (6.15) and (6.16) of [23]). In dimension $d = 1$, this system simply becomes a system of two first-order ODEs. Yet, this system can be even more simplified, as it can be summarized into a particular second-order ODE, whose evolution can be better understood (see [33, 12]). In particular, as already noticed, an important feature is that both solutions to the system of two first-order ODEs and the second-order ODE are periodic, whatever the initial data are.

Before introducing this ODE, we define

$$\mathbb{C}^+ := \{z \in \mathbb{C}, \operatorname{Re} z > 0\}.$$

Proposition 2.1.6. *For any* $\alpha \in \mathbb{C}^+$*, consider the ordinary differential equation*

$$\ddot{r}_\alpha = \frac{1}{r_\alpha^3} - \frac{2\lambda}{r_\alpha}, \qquad r_\alpha(0) = \operatorname{Re}\alpha =: \alpha_r, \qquad \dot{r}_\alpha(0) = \operatorname{Im}\alpha =: \alpha_i.$$

It has a unique solution $r_\alpha \in \mathcal{C}^\infty(\mathbb{R})$ *with values in* $(0,\infty)$*. Moreover, it is periodic.*

We can now properly define the breathers in dimension $d = 1$.

Proposition 2.1.7 (Breathers for logNLS in dimension 1)**.** *For any* $\alpha \in \mathbb{C}^+$*, set*

$$u^\alpha(t,x) := \sqrt{\frac{\alpha_r}{r_\alpha(t)}} \exp\Big[\frac{1}{2} - i\Phi^\alpha(t) - \frac{x^2}{2 r_\alpha(t)^2} + i \frac{\dot{r}_\alpha(t)}{r_\alpha(t)} \frac{x^2}{2}\Big], \qquad t, x \in \mathbb{R},$$

where

$$\Phi^\alpha(t) := \frac{1}{2} \int_0^t \frac{1}{r_\alpha(s)^2} \, \mathrm{d}s + \lambda \int_0^t \ln \frac{r_\alpha(s)}{\alpha_0} \, \mathrm{d}s - \lambda t.$$

Then u^α *is solution to 2.1.1 in dimension* $d = 1$*.*

We emphasize that these solutions to 2.1.1 are periodic in time up to a time-depending complex number of modulus 1; to be more precise, $u^\alpha \exp\Big[-i\Phi^\alpha(t)\Big]$ (with $\Phi^\alpha(t)$ real) is periodic. Therefore, excluding the case $\alpha = (2\lambda)^{-\frac{1}{2}}$ which is the Gausson, this explains why we call those solutions *breathers*. It is worth pointing out that other solutions to 2.1.1 in dimension 1, to be more precise

$$G_{\omega,x_0}(t,x) := \exp\Big[-2it\lambda\omega + \omega - \lambda(x - x_0)^2\Big], \qquad t, x \in \mathbb{R},$$

for any $\omega, x_0 \in \mathbb{R}$, are known to be periodic in time, but their shape does not evolve ($|G_{\omega,x_0}|$ is independent of time) contrary to these breathers.

From these results, the behaviour of these breathers and their shapes are in complete correlation with the knowledge of r_α and its evolution in time. Therefore, it is interesting to study r_α more deeply. In particular, we can study its period, more precisely its evolution with respect to the initial data of r_α. Indeed, it cannot be easily computed (its expression is rather complex, with an integral which cannot be explicitly computed), and one can wonder what regularity with respect to the parameters it has. Still, some approximations are available in two cases: the case of small oscillations (around $(2\lambda)^{-\frac{1}{2}}$) and the case of "big" oscillations. An interesting feature in the latter was found as it appears that the period can become very large. We also postpone a discussion about the behaviour of u^α in this case in Section 2.2.3.

Theorem 2.1.8. *The period* T_α *of* r_α *is continuous with respect to* α*. In addition, it satisfies* $T_\alpha \longrightarrow \frac{\pi}{\sqrt{2}\,\lambda}$ *when* $\alpha \to (2\lambda)^{-\frac{1}{2}}$ *and* $T_\alpha \sim \sqrt{\frac{\pi}{\lambda}}\, \alpha_r \exp\Big[\frac{\alpha_i^2}{4\lambda} + \frac{1}{4\lambda\alpha_r^2}\Big]$ *when* $\alpha \to \infty$ *or* $\alpha_r \to 0$*.*

Those breathers can also be generalized in higher dimension $d \geq 2$. Indeed, as already noticed in [23], if the Gaussian initial data 2.1.3 is such that $\operatorname{Re} A_{\text{in}}$ and $\operatorname{Im} A_{\text{in}}$ commute, then $\operatorname{Re} A(t)$ and $\operatorname{Im} A(t)$ commute for any $t \in \mathbb{R}$ (since $[\operatorname{Re} A(t), \operatorname{Im} A(t)]$ is constant) and can be orthogonally co-diagonalized by the same time-independent orthogonal basis. To this aim, define the orthogonal group $\mathcal{O}_d(\mathbb{R})$ and set

$$S_d(\mathbb{C})^{\operatorname{Re}+} := \{A \in S_d(\mathbb{C}); \operatorname{Re} A \text{ is positive definite }\},$$
$$S_d(\mathbb{C})^{\operatorname{Re}++} := \left\{A \in S_d(\mathbb{C})^{\operatorname{Re}+}; \operatorname{Re} A \text{ and } \operatorname{Im} A \text{ commute}\right\}.$$

Proposition 2.1.9 (Characterization of breathers for logNLS)**.** *Let $d \in \mathbb{N}^*$ and $A \in S_d(\mathbb{C})^{\operatorname{Re}++}$. Set*

$$u_{\text{in}}^A(x) = \exp\Big[\frac{d}{2} - x^\top A x\Big], \qquad x \in \mathbb{R}^d,$$

and u^A the solution to 2.1.1 with initial data u_{in}^A.

Then there exists $R \in \mathcal{O}_d(\mathbb{R})$ and $\alpha_1, \dots, \alpha_d \in \mathbb{C}^+$ such that for all $x \in \mathbb{R}^d$,

$$u^A(t, Rx) = u^{\alpha_1}(t, x_1)\, u^{\alpha_2}(t, x_2) \dots u^{\alpha_d}(t, x_d).$$

However, in dimension $d \geq 2$, those generalized breathers are only a particular case among the more generalized class of Gaussian solutions of the form 2.1.4. It has also already been pointed out (in [23]) that the positive definiteness of $\operatorname{Re} A(t)$ is preserved and the oscillations are bounded in amplitude. Even more, the spectrum of $\operatorname{Re} A(t)$ can be bounded by below by a positive time-independent constant a_{inf} but also by above, and along with the relation between $\operatorname{Re} A(t)$ and $|b(t)|$,

$$|b(t)| = \left(\frac{\det \operatorname{Re} A(0)}{\det \operatorname{Re} A(t)}\right)^{\frac{1}{4}},$$

we see that some features of the breathers remain true even though the periodicity is lost.

We denote those Gaussian solutions to 2.1.1 by taking care of the invariants and the scaling factor: for any $A_{\text{in}} \in S_d(\mathbb{C})^{\operatorname{Re}+}$, $x_0, v \in \mathbb{R}^d$, $\omega, \theta \in \mathbb{R}$, we set

$$G^d_{A_{\text{in}},\omega,x_0,v,\theta}(t,x) := \exp\left[i\left(\theta + 2\lambda\omega t - v\cdot x + \frac{|v|^2}{2}t\right) + \omega\right] u^{A_{\text{in}}}(t, x - x_0 - vt), \tag{2.1.5}$$

where $u^{A_{\text{in}}}$ is the solution to 2.1.1 with initial data 2.1.3.

Nonlinear superposition. In the context of solitons or breathers for a non-linear dispersive equation, an important question is the understanding of the interactions between them for an initial data made of the sum of several decoupled solitons or breathers. The qualitative information which come from this study should allow to better understand the dynamics and behaviours induced by the equation and its flow. One of the usual related topics is the problem of the existence of multi-soliton solution, i.e. a solution which converges (in some sense) to the sum of solitons when $t \to \infty$, and of its stability in order to investigate whether or not they are generic objects for the dynamics of this equation.

The inverse scattering transform method was the first method used to construct multi-solitons for NLS equation [161]. However, such a method is restricted to equations which are completely integrable (like the Korteweg-de Vries equation and the cubic nonlinear Schrödinger equation in dimension 1). Another method, using *energy techniques*, meaning that it relies on the use of the second variation of the energy as a Lyapunov functional to control the difference of a solution with the soliton sum, appeared in the early 2000 [117] and has been used a lot since (e.g. [116, 118, 55, 53]).

Known results on the question of stability of multisolitary wave solutions are based on asymptotic stability, which means that the solution converges (in some sense) as $t \to \infty$ to the sum of several solitary waves. It is relatively natural to expect that, at least when the interactions are local and the composing solitons are exponentially decaying at infinity, a multi-soliton will be orbitally stable if all the composing solitons are orbitally stable. For some equations, like the generalized Korteweg–de Vries (gKdV) and the nonlinear Schrödinger equation (NLS), the orbital stability in H^1 of this sum has been shown under an assumption of flatness of the nonlinearity, and a sufficient relative speed between the solitons in the case of NLS (see for instance [117, 118] and the references in there). For instance, the KdV equation is also known for having special explicit solutions, called N-solitons ($N \geq 2$), corresponding to the superposition of N traveling waves with different speeds that interact and then remain unchanged after interaction and behaving asymptotically in large time as the sum of N traveling waves (see [122]). Those N-solitons are also orbitally stable in H^1 with a more precise description of the asymptotic stability (see [117]).

On the other hand, as soon as one of the solitons of the sum is unstable, the multi-soliton constructed for this sum is expected to be unstable. Such a result has been proved by R. Côte and S. Le Coz [53] for the subcritical NLS equation with instability in H^1. We can also cite [43, 82, 125] for partial results in the L^2-subcritical case and [81, 100, 123, 124] for results on instability with a supercritical nonlinearity.

However, even if it is unstable, a multi-soliton may still exist for this sum with (at least) one unstable soliton as soon as the relative speed is large enough ([53]). Thus, even though this multi-soliton should be unstable, a solution with initial data close to this multi-soliton (for instance equal to the sum of solitons) will remain close to it for some time, which should be long if the solitons in the sum get away from each other, since the interactions between them become smaller and smaller as the distance between the solitons increases (remember that the solitons vanish at infinity, and often decrease exponentially). Such a study can even be generalized to a sum of *standing* solitons (meaning that the minimum relative speed is 0), as long as they are far away from each other.

One can also wonder if breathers are stable, or also if multi-breathers exist and are stable too. Indeed, some breathers are known to be stable, for instance in H^1 for the mKdV equation [5, 4]. The question of existence and stability of multi-breathers for mKdV have also been answered in these articles (see also [46]). Alternatively, the previous problem of *nonlinear superposition* (i.e. how long a solution with a sum of breathers as initial data will remain close to this sum) seems an interesting first question in this way.

For 2.1.1, we have a large class of Gaussian functions solution which includes solitons (Gaussons) and breathers. However, the only thing we know about them is that the Gausson is orbitally stable. To be able to understand further the behaviour of this equation, some numerical methods have been developed by W. Bao, R. Carles, C. Su and Q. Tang [12, 13]. In the latter, some numerical simulations have been performed and very interesting and new features have been found. For instance, some of the behaviours found in these simulations along with the orbital stability of the Gausson suggest not only the existence of multi-solitons [68], but also their stability.

But we will mostly focus in particular on the first two simulations in Fig. 4.5 of this article, who falls into our study with $\lambda = 1$. Beginning with two Gaussons whose distance from each other is 10 (for the first simulation) or 6 (for the second), the behaviour of the solution is rather different. For the former, it gives the impression that the interactions between the two Gaussons are so small that nothing seems to happen: the numerical solution has the same form at any time as the initial data and remains almost constant for a very large time. On the other hand, for the latter, the interactions between the Gaussons make them move closer from each other, very slowly at first but then faster and faster, until they cross each other at time $t = 13.6$ without (almost) any change in their form, except two little structures which go to infinity on both sides. From this moment, we observe an almost periodic behaviour: the two "Gaussons" oscillate, crossing each other almost regularly, whereas some other little structures appear (less and less regularly) and go to infinity.

Therefore, a huge change in the behaviour of the solution is seen from a small change in the distance between the two Gaussons. In particular, the first simulation seems to show that the structure is rather stable, which is very surprising since no standing multi-solitons have been proved to be stable or even exist for any equation (to the best of our knowledge). Yet, the simulation may not reflect the real solution for large times, thus we will simply say that the solution remain close to the sum of the two Gaussons for a large time.

The main result of this paper is a partial result about this observation, with an estimate of the L^2 distance between the sum of solitons/breathers/Gaussian solutions of 2.1.1 and the solution of 2.1.1 with the previous sum as initial data.

Theorem 2.1.10 (Nonlinear superposition principle for logNLS). *Let $d \in \mathbb{N}^*$. There exists $C_d > 0$ such that the following holds. Consider $\lambda > 0$, $N \in \mathbb{N}^*$ and take $x_k \in \mathbb{R}^d$, $A^k_{\text{in}} \in S_d(\mathbb{C})^{\mathrm{Re}+}$, $\omega_k \in \mathbb{R}$ and $\theta_k \in \mathbb{R}$ for $k = 1, \ldots, N$ and $v \in \mathbb{R}$. Let u the solution to 2.1.1 with initial data $u_{\text{in}}(x) := \sum G^d_{A^k_{\text{in}},\omega_k,x_k,v,\theta_k}(0, x)$ for any $x \in \mathbb{R}^d$. Define $A_k(t)$ provided by Proposition 2.1.5 for each A^{in}_k and set $G(t) := \sum G^d_{A_k,\omega_k,x_k,v,\theta_k}(t)$ and*

$$\tau_- := \inf_{t,k,j} \sigma(\mathrm{Re}\, A_k(t)), \qquad \tau_+ := \sup_{t,k,j} \sigma(\mathrm{Re}\, A_k(t)),$$

where $\sigma(M)$ designates the spectrum of a matrix M.

Then $0 < \tau_- \le \tau_+ < \infty$ and there exist $\varepsilon_0 > 0$ depending only on $\delta\omega := \max_k |\omega_k - \omega_{k+1}|$, τ_-, τ_+ and N such that if

$$\varepsilon := \left(\min_k |x_{k+1} - x_k|\right)^{-1} < \varepsilon_0,$$

then for all $t \ge 0$,

$$\|u(t) - G(t)\|_{L^2(\mathbb{R}^d)} \le C_d N^{\frac{3}{2}} \frac{\lambda\, \tau_+}{\varepsilon^{\frac{d}{2}+1}\sqrt{\tau_-}} \exp\left[-\frac{\tau_-}{4\varepsilon^2} + \max_j \omega_j + 2\lambda t\right].$$

Remark 2.1.11. If the real and imaginary parts of A_k commute, then Proposition 2.1.9 applies and gives $\alpha^k_1, \ldots, \alpha^k_d$ coming from A_k for all $k = 1, \ldots, N$. Hence, it is easy to prove that

$$\tau_- = \frac{1}{2} \min_{t,k,j} r_{\alpha^k_j}(t)^{-2}, \qquad \tau_+ = \frac{1}{2} \max_{t,k,j} r_{\alpha^k_j}(t)^{-2}.$$

Furthermore, if all the A_ks are equal to $\lambda\, I_d$, then $\tau_- = \tau_+ = \lambda$ and the inequality becomes

$$\left\|u(t) - \sum G^d_{A_k,\omega_k,x_k,v,\theta_k}(t)\right\|_{L^2(\mathbb{R}^d)} \le C_d N^{\frac{3}{2}} \frac{\lambda^{\frac{3}{2}}}{\varepsilon^{\frac{d}{2}+1}} \exp\left[-\frac{\lambda}{4\varepsilon^2} + \max_j \omega_j + 2\lambda t\right],$$

for all $t \geq 0$.

This result gives an interesting time during which the solution u remains close to $G := \sum G^d_{A_k,\omega_k,x_k,v,\theta_k}$ when the minimum distance between the solitons/breathers/Gaussian solutions is large enough. Indeed, take $\delta > 0$ as small as we want. The previous result say that the inequality

$$\left\| u(t,.) - \sum G^d_{A_k,\omega_k,x_k,v,\theta_k}(t,.) \right\|_{L^2(\mathbb{R}^d)} \leq \delta$$

holds for all $t \in [0, t_\delta]$ where

$$t_\delta := \frac{\tau_-}{8\lambda\,\varepsilon^2} - \frac{\omega}{2\lambda} + \frac{\frac{d}{2}+1}{2\lambda}\ln\varepsilon + \frac{1}{2\lambda}\ln\left[\frac{\delta\sqrt{\tau_-}}{C_d N^{\frac{3}{2}}\lambda\tau_+}\right],$$

with $\omega := \max_j \omega_j$, as soon as ε is small enough. Thus, if we fix everything except $(x_k)_k$ and then take a sequence of family $(x^n_k)_{1\leq k\leq N, n\in\mathbb{N}^*}$ such that

$$\varepsilon_n := \left(\min_k |x^n_{k+1} - x^n_k|\right)^{-1} \underset{n\to\infty}{\longrightarrow} 0,$$

the resulting t^n_δ can be expanded as

$$t^n_\delta = \frac{\tau_-}{8\lambda\,\varepsilon_n^2} + \frac{\frac{d}{2}+1}{2\lambda}\ln\varepsilon_n + O(1) \sim \frac{\tau_-}{8\lambda\,\varepsilon_n^2}.$$

For instance, if all the A_ks are equal to $\lambda\, I_d$, then $\tau_- = \lambda$ and we get at first order

$$t^n_\delta \sim \frac{1}{8\varepsilon_n^2}.$$

It is interesting to see that this time is in square of the minimal distance between the Gaussian functions which are in the sum. This can explain the difference we have seen in the previous two numerical examples of [13] and the fact that a rather small change in the distance between the two Gaussons imply a bigger change in the time until which the solution remains close to the sum.

2.1.4. Outline of the paper

Section 2.2 is devoted to the study of the solution to 2.1.1 with Gaussian initial data. We recall in there the way to get explicit solutions (as already proved in [23, 33, 12]) which leads to Propositions 2.1.6 and 2.1.7, and study more carefully the behaviour of r_α, in particular its period by proving Theorem 2.1.8. We also finish by a discussion about how the behaviour of r_α affects the behaviour of u^α (in Section 2.2.3) and by a brief proof of Proposition 2.1.9. In Section 2.3, we prove Theorem 2.1.10. The proof rely on a computation inspired from the energy estimate in L^2 (available thanks to Lemma 2.3.1) and on Lemma 2.3.2, whose proof takes most of the Section.

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2.2. PROPAGATION OF GAUSSIAN DATA

In this section, we prove Propositions 2.1.7 and 2.1.9 and we describe more precisely the behaviour of r_α, and in particular its period. As already noticed in [23] and more rigorously analyzed in [33, 12], an important feature of 2.1.1 is that the evolution of initial Gaussian data remains Gaussian. In particular, the case $d = 1$ is interesting since we obtain a system of 2 ODEs which can be reduced into a single ODE.

2.2.1. From 2.1.1 to ordinary differential equations

We seek a solution to 2.1.1 (in dimension $d = 1$) under the form:

$$u(t) = b(t)\exp\Big(\frac{1}{2} - \frac{1}{2}\mu(t)x^2\Big), \qquad t, x \in \mathbb{R},$$

where $\mu(t), b(t) \in \mathbb{C}$ with $\operatorname{Re}\mu(t) > 0$ for all t. We can also assume $b(0) = 1$ thanks to Remark 2.1.1.

Remark 2.2.1. We recall the expression of the (one-dimensional) Gausson

$$\exp\Big(\frac{1}{2} - \lambda x^2\Big).$$

This explains why we took this form: $b(t) = 1$ and $\mu(t) = 2\lambda$ is therefore a solution.

It has already been shown in [23] that b takes the form

$$b(t) = \left(\frac{\mu_r(t)}{\mu_r(0)}\right)^{\frac{1}{4}} e^{i\phi(t)},$$

where $\mu_r := \operatorname{Re}\mu$ and $\phi(t)$ is given by

$$\phi(t) = -\frac{1}{2}\int_0^t \mu_r(s)\,\mathrm{d}s + \frac{\lambda}{2}\int_0^t \ln\frac{\mu_r(s)}{\mu_r(0)}\,\mathrm{d}s + \lambda t,$$

and the evolution of μ ($\mu(t) \in \mathbb{C}$) is driven by this ordinary differential equation

$$-i\dot{\mu}(t) + \mu(t)^2 = 2\lambda\mu_r(t).$$

The latter can actually be rewritten in a simpler way: indeed, μ can be expressed as (see [33])

$$\mu = \frac{1}{r^2} - i\frac{\dot{r}}{r}, \tag{2.2.1}$$

where r is real and satisfies the ODE

$$\ddot{r} = \frac{1}{r^3} - \frac{2\lambda}{r}. \tag{2.2.2}$$

Remark 2.2.2. The initial data of r are given by the initial data of μ through 2.2.1, thus the two degrees of freedom can be put back on r. Indeed, $r(0) = (\mu_r(0))^{-\frac{1}{2}}$ can take any positive value whereas $\dot{r}(0) = -(\mu_r(0))^{-\frac{1}{2}}\,\mu_i(0)$ can take any real value independently. Thus, for any $\alpha = \alpha_r + i\alpha_i$ such that $\alpha_r > 0$ and $\alpha_i \in \mathbb{R}$, we will denote by r_α the solution to 2.2.2 with initial data $r_\alpha(0) = \alpha_r$ and $\dot{r}_\alpha(0) = \alpha_i$.

Hence, for any $\alpha = \alpha_r + i\alpha_i$ such that $\alpha_r > 0$ and $\alpha_i \in \mathbb{R}$ (i.e. $\alpha \in \mathbb{C}^+$),

$$u^\alpha(t) := \sqrt{\frac{\alpha_r}{r_\alpha(t)}}\exp\Big(i\phi^\alpha(t) + \frac{1}{2} - \frac{1}{2r_\alpha(t)^2}x^2 + i\frac{\dot{r}_\alpha(t)}{r_\alpha(t)}\frac{x^2}{2}\Big), \qquad t, x \in \mathbb{R},$$

where

$$\phi^\alpha(t) = -\frac{1}{2}\int_0^t \frac{1}{r_\alpha(s)^2}\,\mathrm{d}s - \lambda\int_0^t \ln\frac{r_\alpha(s)}{\alpha_r}\,\mathrm{d}s + \lambda t,$$

is solution to 2.1.1.

Remark 2.2.3. In particular, in the continuity of Remark 2.2.1, the Gausson 2.1.2 is u^α for $\alpha = (2\lambda)^{-\frac{1}{2}}$. Indeed, we can easily prove that $r_\alpha(t) = (2\lambda)^{-\frac{1}{2}}$ and $\phi^\alpha(t) = 0$ for all $t \in \mathbb{R}$ with those initial data.

2.2.2. Study of r_α

First of all, we rescale the equation in order to make the 2λ factor disappear. Indeed, if we define

$$\tau_\gamma(t) := \sqrt{2\lambda}\, r_\alpha\Big(\frac{t}{2\lambda}\Big), \qquad t \in \mathbb{R},$$

with $\gamma := \sqrt{2\lambda}\alpha_r + i\frac{\alpha_i}{\sqrt{2\lambda}}$, then τ_γ satisfies

$$\ddot{\tau}_\gamma = \frac{1}{\tau_\gamma^3} - \frac{1}{\tau_\gamma}, \qquad \tau_\gamma(0) = \gamma_r := \operatorname{Re}\gamma > 0, \qquad \dot{\tau}_\gamma(0) = \gamma_i := \operatorname{Im}\gamma. \tag{2.2.3}$$

Thus, there remains to study τ_γ instead of r_α.

First, we should prove that τ_γ is well defined. For any $\gamma \in \mathbb{C}^+$, the Cauchy-Lipschitz theorem gives a local definition of τ_γ. However, since $f(x) := x^{-3} - x^{-1} \to +\infty$ when $x \to 0^+$, we need to check that $\tau_\gamma(t)$ never touches 0 in finite time in order to prove that $\tau_\gamma(t)$ is defined for all $t \in \mathbb{R}$. Such a result can be proved thanks to a conserved quantity. Indeed, 2.2.3 has an Hamiltonian structure of the form:

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q},$$

where

$$H(p, q) = \frac{1}{2}p^2 + F(q),$$

with

$$F(q) = \frac{1}{2q^2} + \ln q$$

an anti-derivative of f. In particular, H is conserved by the flow of the equation, hence there holds

$$E(\tau_\gamma) := 2\,H(\dot{\tau}_\gamma, \tau_\gamma) = (\dot{\tau}_\gamma)^2 + \frac{1}{(\tau_\gamma)^2} + 2\ln\tau_\gamma = E_\gamma, \tag{2.2.4}$$

where

$$E_\gamma := \gamma_i^2 + \frac{1}{(\gamma_r)^2} + 2\ln\gamma_r. \tag{2.2.5}$$

We emphasize that

$$F(q) \to +\infty, \qquad \text{when either } q \to 0 \text{ or } q \to +\infty. \tag{2.2.6}$$

Hence, it is easy to prove that $\tau_\gamma(t)$ never touches 0 (and has actually a positive lower bound) and also has an upper bound.

Proposition 2.2.4. *For any $\gamma \in \mathbb{C}^+$, $\tau_\gamma \in \mathcal{C}^\infty(\mathbb{R})$ and there holds for all $t \in \mathbb{R}$*

$$\frac{1}{1+\sqrt{E_\gamma - 1}} \le \tau_\gamma(t) \le \exp\frac{E_\gamma}{2},$$

where $E_\gamma \ge 1$ is defined by 2.2.5.

Proof. In view of the previous remark, proving the lower bound readily leads to the fact that $\tau_\gamma \in \mathcal{C}^\infty(\mathbb{R})$ since f is $\mathcal{C}^\infty$ on $(0,\infty)$. To prove this lower bound, we use 2.2.4. Indeed, for any $t \in I$ where I is the (maximal) interval of definition for τ_γ, there holds

$$2\ln\tau_\gamma(t) = -2\ln\frac{1}{\tau_\gamma(t)} \ge -2\Big(\frac{1}{\tau_\gamma(t)} - 1\Big),$$

where we used the fact that for all $x > 0$, $\ln x \le x - 1$. Thus, plugging this inequality into 2.2.4 yields

$$\frac{1}{(\tau_\gamma)^2} - 2\Big(\frac{1}{\tau_\gamma(t)} - 1\Big) \le E_\gamma, \qquad \text{i.e.} \qquad \Big(\frac{1}{\tau_\gamma(t)} - 1\Big)^2 \le E_\gamma - 1.$$

In particular, there also holds

$$\frac{1}{\tau_\gamma(t)} - 1 \le \sqrt{E_\gamma - 1},$$

and then the lower bound for $\tau_\gamma(t)$ and the fact that τ_γ is defined on $\mathbb{R}$ readily follow. On the other hand, there also holds thanks to 2.2.4

$$2\ln\tau_\gamma \le E_\gamma,$$

which leads to the upper bound. □

Actually, the behaviour of τ_γ can be better characterized. Indeed, we also emphasize that

$$f(1) = 0, \qquad f(x) < 0 \quad \forall x > 1, \qquad f(x) > 0 \quad \forall x \in (0,1).$$

Thus, the phase portrait (drawn in Figure 2.2.1) is rather simple and looks like that of Lotka-Volterra or prey-predator system. In particular, the trajectories are actually the level sets of $H(p,q)$. Then they describe closed Jordan curves symmetric to the x-axis that surround the point $(1,0)$ in the phase portrait, represented by

$$p = \pm\sqrt{E_\gamma - 2F(q)} \qquad \text{for} \quad \gamma_- \le q \le \gamma_+,$$

where $0 < \gamma_- < 1 < \gamma_+$ are the two solutions to the equation $2F(q) = E_\gamma$ with unknown q. These values are uniquely determined because of the strict monotonicity of F in the intervals $(0,1)$ and $(1,\infty)$ and $F(1) = 1 < E_\gamma$ and 2.2.6. Thus, similar arguments as for the Lotka-Volterra system can be applied, and then lead to the following Proposition.

Proposition 2.2.5. *For any $\gamma \in \mathbb{C}^+$, τ_γ is periodic. Moreover, for any $\gamma \in \mathbb{C}^+ \setminus \{1\}$, we have $E_\gamma > 1$ and therefore the period T_γ is given by*

$$T_\gamma = 2\int_{\gamma_-}^{\gamma_+} \frac{\mathrm{d}x}{\sqrt{E_\gamma - \frac{1}{x^2} - 2\ln x}}, \tag{2.2.7}$$

where γ_- (resp. γ_+) is the only solution on $(0,1)$ (resp. $(1,\infty)$) of

$$\frac{1}{x^2} + 2\ln x = E_\gamma.$$

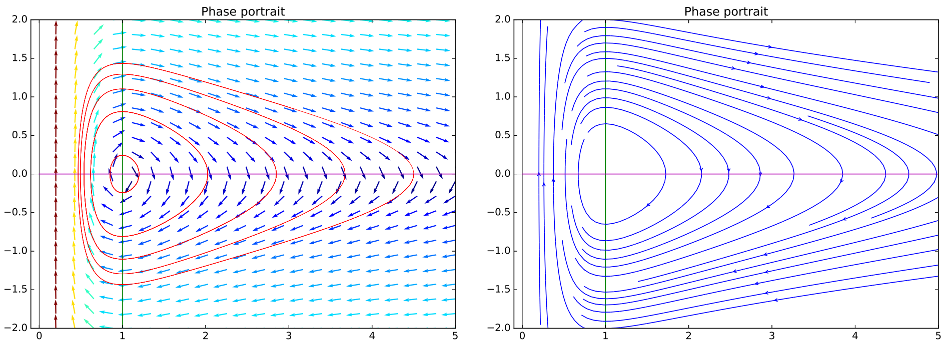

Figure 2.2.1 – Phase portraits. At the left-hand side, the arrows give the direction of the flow; the green and magenta lines are the horizontal and vertical isoclines respectively; the red lines are some trajectories, which fits the level set of $H(p,q)$. At the right-hand side, a part of some trajectories are drawn along with the isoclines again.

Proof. For a more precise proof, we refer to the Chapters 3.VII. and 11.X. of [146], by adapting it for a smaller interval than $(-\infty,+\infty)$. This is also a re-writing of equation (6.28) of [23]. □

Now, we prove Theorem 2.1.8, starting by the continuity of the period with respect to the parameters. With the previous result, we know that the definition of γ_- and γ_+ depends only on $E_\gamma > 1$, hence the period depends only on E_γ, which is continuous with respect to $\gamma \in \mathbb{C}^+$ thanks to 2.2.5. Therefore, we only need to prove the continuity of T_γ with respect to $E_\gamma > 1$.

Proposition 2.2.6. *T_γ is continuous with respect to $E_\gamma > 1$, hence also with respect to $\gamma \in \mathbb{C}^+ \setminus \{1\}$.*

First, we shall prove the regularity of γ_- and γ_+ with respect to E_γ (hence also to γ).

Lemma 2.2.7. *γ_- and γ_+ are $\mathcal{C}^\infty$ with respect to $E_\gamma \in (1,\infty)$.*

Proof. The definition of γ_- and γ_+ leads to those two properties:

$$\begin{cases} F(q) - \frac{E_\gamma}{2} = 0 \\ q \in (0,1) \end{cases} \iff q = \gamma_-(E_\gamma), \qquad \begin{cases} F(q) - \frac{E_\gamma}{2} = 0 \\ q > 1 \end{cases} \iff q = \gamma_+(E_\gamma).$$

Moreover, we know that F is $\mathcal{C}^\infty$ on $(0,\infty)$ and that for all $q \in (0,\infty) \setminus \{1\}$, there holds

$$F'(q) = f(q) \neq 0.$$

Therefore the conclusion readily follows from the implicit function theorem. □

Proof of Proposition 2.2.6. First, we cut the integral in 2.2.7 into 2 :

$$\int_{\gamma_-}^{\gamma_+} \frac{\mathrm{d}x}{\sqrt{E_\gamma - \frac{1}{x^2} - 2\ln x}} = \int_{\gamma_-}^{1} \frac{\mathrm{d}x}{\sqrt{E_\gamma - \frac{1}{x^2} - 2\ln x}} + \int_{1}^{\gamma_+} \frac{\mathrm{d}x}{\sqrt{E_\gamma - \frac{1}{x^2} - 2\ln x}}. \tag{2.2.8}$$

Thanks to this equality, we prove the continuity of T_γ by proving the continuity of the two integrals in the right-hand side. For example, for the latter, there holds

$$\begin{aligned} \int_1^{\gamma_+} \frac{\mathrm{d}x}{\sqrt{E_\gamma - \frac{1}{x^2} - 2\ln x}} &= \int_1^{(\gamma_+)^2} \frac{\mathrm{d}y}{2\sqrt{y}\sqrt{E_\gamma - \frac{1}{y} - \ln y}} \\ &= \int_0^{\delta_+} \frac{\mathrm{d}z}{2\sqrt{1+z}\sqrt{E_\gamma - \frac{1}{1+z} - \ln(1+z)}} \\ &= \int_0^1 \frac{\delta_+ \,\mathrm{d}x}{2\sqrt{1+\delta_+ x}\sqrt{E_\gamma - \frac{1}{1+\delta_+ x} - \ln(1+\delta_+ x)}}, \end{aligned} \tag{2.2.9}$$

where $\delta_+ = (\gamma_+)^2 - 1 > 0$. In particular, δ_+ is continuous with respect to E_γ, therefore the integrand is continuous with respect to E_γ. Moreover, setting $g_{\delta_+}(x) = \frac{1}{1+\delta_+ x} - \ln(1+\delta_+ x)$, there holds

$$g_{\delta_+}(1) = E_\gamma, \qquad g'_{\delta_+}(x) = (\delta_+)^2 \frac{x}{(1+\delta_+ x)^2} \geq \frac{(\delta_+)^2}{(1+\delta_+)^2} x \qquad \forall x \in [0,1].$$

Thus, there holds for all $x \in [0,1]$

$$E_\gamma - g_{\delta_+}(x) \geq \frac{(\delta_+)^2}{(1+\delta_+)^2} \int_x^1 y \, \mathrm{d}y = \frac{(\delta_+)^2}{2(1+\delta_+)^2}(1-x^2) \geq \frac{(\delta_+)^2}{2(1+\delta_+)^2}(1-x).$$

Hence, there also holds

$$\frac{\delta_+}{2\sqrt{1+\delta_+ x}\sqrt{E_\gamma - \frac{1}{1+\delta_+ x} - \ln(1+\delta_+ x)}} \leq \frac{1+\delta_+}{\sqrt{2(1-x)}}.$$

The continuity of the right-hand side of 2.2.9 readily follows from the theorem of continuity under integral sign. In the same way, the first integral in the right-hand side of 2.2.8 can be transformed into

$$\begin{aligned}
\int_{\gamma_-}^1 \frac{\mathrm{d}x}{\sqrt{E_\gamma - \frac{1}{x^2} - 2\ln x}} &= \int_{(\gamma_-)^2}^1 \frac{\mathrm{d}y}{2\sqrt{y}\sqrt{E_\gamma - \frac{1}{y} - \ln y}} \\
&= \int_1^{\frac{1}{(\gamma_-)^2}} \frac{\mathrm{d}z}{2z^{\frac{3}{2}}\sqrt{E_\gamma - z + \ln z}} \\
&= \int_0^{\delta_-} \frac{\mathrm{d}z}{2(1+z)^{\frac{3}{2}}\sqrt{E_\gamma - (1+z) + \ln(1+z)}} \\
&= \int_0^1 \frac{\delta_- \, \mathrm{d}z}{2(1+\delta_- z)^{\frac{3}{2}}\sqrt{E_\gamma - (1+\delta_- z) + \ln(1+\delta_- z)}},
\end{aligned}$$

where $\delta_- = \frac{1}{(\gamma_-)^2} - 1 > 0$. Setting $h_{\delta_-}(x) \coloneqq 1 + \delta_- x - \ln(1+\delta_- z)$, there holds in the same way:

$$h_{\delta_-}(1) = E_\gamma, \qquad h'_{\delta_-}(x) = (\delta_-)^2 \frac{x}{1+\delta_- x} \geq \frac{(\delta_-)^2}{1+\delta_-} x \qquad \forall x \in [0,1].$$

Hence similar arguments can be applied here and yield the continuity of the previous integral. Thus, we obtain the continuity of T_γ with respect to E_γ, hence also with respect to γ since E_γ is continuous with respect to γ. □

Now that the continuity of the period with respect to the parameters is proved, and since it is impossible to get a simpler expression of this period, we shall find some approximations. In particular, the question (i) of the Chapter 11.XI. of [146] is also interesting since, along with the fact that $f'(1) = -2$, it gives the limit of the period of τ_γ when γ goes to 1. We can also cite [126] for a proof of this result.

Proposition 2.2.8. *When $\gamma \to 1$, $T_\gamma \to \sqrt{2}\,\pi$.*

On the other hand, another interesting question is the period in the case of big oscillations. As one may think from the phase portrait (Fig 2.2.1) and even more from the big trajectories (Fig 2.2.2), a small increase in the initial energy (the energy of the trajectories in Fig 2.2.2 goes from 4 to 6) induces a big increase for the maximum of τ_γ (from 20 to 50) and for the period. To prove this behaviour, explicit computations and inequalities are required and yield the following result.

Proposition 2.2.9. *When $\gamma \to \infty$ or $\operatorname{Re}\gamma \to 0$,*

$$T_\gamma \sim \sqrt{2\pi} \exp \frac{E_\gamma}{2}.$$

Proof. First, we emphasize that the condition $\gamma \to \infty$ or $\operatorname{Re}\gamma \to 0$ is equivalent to the simpler condition $E_\gamma \to +\infty$, and therefore also to the facts that $\gamma_- \to 0$ and $\gamma_+ \to +\infty$. To be more precise for γ_+, there holds

$$2\ln\gamma_+ + o(1) = E_\gamma, \qquad \text{i.e.} \qquad \gamma_+ \sim \exp\frac{E_\gamma}{2}.$$

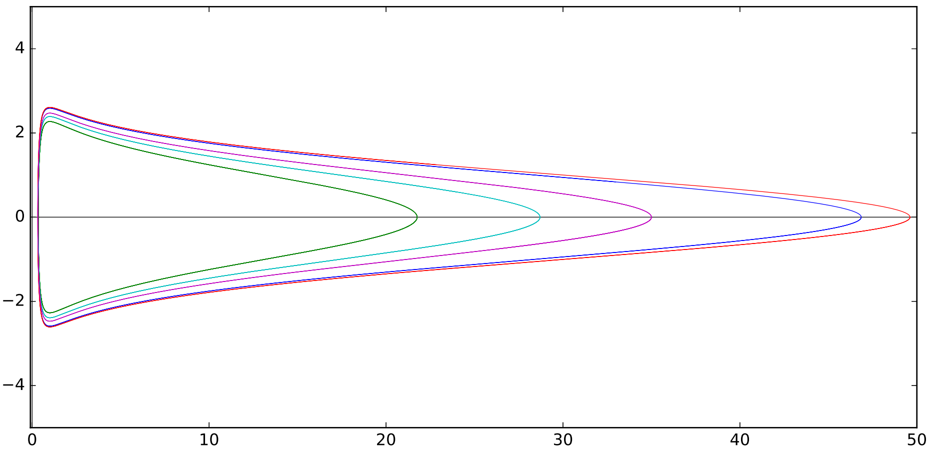

Figure 2.2.2 – Plot of some trajectories of τ_γ in the phase space for 5 real γ between 10 and 50.

Then we cut the integral in 2.2.7 in two: before 1 and after 1. First,

$$\begin{aligned}\int_{\gamma_-}^{1}\frac{\mathrm{d}x}{\sqrt{E_\gamma-\frac{1}{x^2}-2\ln x}}&=\int_{\gamma_-}^{1}\frac{\mathrm{d}x}{\sqrt{\frac{1}{(\gamma_-)^2}-\frac{1}{x^2}-2\ln\frac{x}{\gamma_-}}}\\&=\gamma_-\int_1^{\frac{1}{\gamma_-}}\frac{\mathrm{d}y}{\sqrt{\frac{1}{(\gamma_-)^2}\left(1-\frac{1}{y^2}\right)-2\ln y}}\\&=(\gamma_-)^2\int_1^{\frac{1}{\gamma_-}}\frac{\mathrm{d}y}{\sqrt{1-\frac{1}{y^2}-2(\gamma_-)^2\ln y}}\\&\le(\gamma_-)^2\int_1^{\frac{1}{\gamma_-}}\frac{\mathrm{d}y}{\sqrt{1-\frac{1}{y^2}-2(\gamma_-)^2(y-1)}}\\&\le(\gamma_-)^2\int_1^{\frac{1}{\gamma_-}}\frac{\mathrm{d}y}{\sqrt{\left(\frac{y+1}{y^2}-2(\gamma_-)^2\right)(y-1)}}.\end{aligned}$$

Moreover, there holds for all $y\in[1,\frac{1}{\gamma_-}]$

$$\frac{y+1}{y^2}\ge\gamma_-+(\gamma_-)^2,$$

so that

$$\begin{aligned}\int_{\gamma_-}^{1}\frac{\mathrm{d}x}{\sqrt{E_\gamma-\frac{1}{x^2}-2\ln x}}&\le\frac{(\gamma_-)^2}{\sqrt{\gamma_--(\gamma_-)^2}}\int_1^{\frac{1}{\gamma_-}}\frac{\mathrm{d}y}{\sqrt{y-1}}\\&\le\frac{\gamma_-}{\sqrt{\frac{1}{\gamma_-}-1}}\left[\frac{1}{2}\sqrt{y-1}\right]_1^{\frac{1}{\gamma_-}}\\&\le\frac{\gamma_-}{2}\longrightarrow 0.\end{aligned}$$

On the other hand, we will prove a lower and an upper bound for the second part of the integral. For the lower bound, we recall that $2F(\gamma_+)=E_\gamma$, so for all $x\in[1,\gamma_+]$

$$E_\gamma-\frac{1}{x^2}=\frac{1}{(\gamma_+)^2}-\frac{1}{x^2}+2\ln\gamma_+\le 2\ln\gamma_+.$$

Therefore,

$$\begin{aligned}\int_1^{\gamma_+} \frac{\mathrm{d}x}{\sqrt{E_\gamma - \frac{1}{x^2} - 2\ln x}} &\geq \int_1^{\gamma_+} \frac{\mathrm{d}x}{\sqrt{2\ln\gamma_+ - 2\ln x}} \\ &\geq \gamma_+ \int_{\frac{1}{\gamma_+}}^1 \frac{\mathrm{d}y}{\sqrt{-2\ln y}}.\end{aligned} \tag{2.2.10}$$

The last integral converges as $\gamma_+ \to \infty$ to

$$\int_0^1 \frac{\mathrm{d}y}{\sqrt{-2\ln y}} = \int_0^\infty \frac{e^{-z}\,\mathrm{d}z}{\sqrt{2z}} = \int_0^\infty \sqrt{2}\,e^{-\zeta^2}\,\mathrm{d}\zeta = \sqrt{\frac{\pi}{2}}.$$

Hence, the right-hand side of 2.2.10 is equivalent to

$$\sqrt{\frac{\pi}{2}}\exp\frac{E_\gamma}{2}.$$

For the upper bound, with a change of variables $x = e^y$ and with $y_+ = \ln\gamma_+ \to +\infty$, we first obtain

$$\int_1^{\gamma_+} \frac{\mathrm{d}x}{\sqrt{E_\gamma - \frac{1}{x^2} - 2\ln x}} = \int_0^{y_+} \frac{e^y\,\mathrm{d}y}{\sqrt{E_\gamma - e^{-2y} - 2y}}.$$

Moreover, using the convexity of $y \mapsto e^{-2y}$, there holds for all $y \in [0, y_+]$

$$e^{-2y} \leq 1 - \frac{y}{y_+}(1 - e^{-2y_+}).$$

Thus, using also the fact that $E_\gamma = e^{-2y_+} + 2y_+$, we obtain

$$\begin{aligned}\int_1^{\gamma_+} \frac{\mathrm{d}x}{\sqrt{E_\gamma - \frac{1}{x^2} - 2\ln x}} &\leq \int_0^{y_+} \frac{e^y\,\mathrm{d}y}{\sqrt{E_\gamma - \left(1 - \frac{y}{y_+}(1 - e^{-2y_+})\right) - 2y}} \\ &\leq \int_0^{y_+} \frac{e^y\,\mathrm{d}y}{\sqrt{\left(2 - \frac{1 - e^{-2y_+}}{y_+}\right)(y_+ - y)}} \\ &\leq \frac{e^{y_+}}{\sqrt{2 - \frac{1 - e^{-2y_+}}{y_+}}} \int_0^{y_+} \frac{e^{-z}\,\mathrm{d}z}{\sqrt{z}} \\ &\leq \frac{\gamma_+}{\sqrt{2 - \frac{1 - e^{-2y_+}}{y_+}}} \int_0^{(y_+)^2} 2e^{-z^2}\,\mathrm{d}z \sim \sqrt{\frac{\pi}{2}}\exp\frac{E_\gamma}{2}.\end{aligned}$$

The conclusion readily follows. □

2.2.3. Discussion on u^α

Those results show an interesting and surprising feature: the "period" of u^α (i.e. the period of r_α) can be very large. Moreover, if we take a Gaussian initial data very concentrated

$$u_0(x) = \exp\Big(\frac{1}{2} - \delta x^2\Big)$$

with $\delta > 0$ large compared to λ, the solution will first disperse, very quickly at the beginning but more and more slowly, until a time when this behaviour turns round. Then the solution will re-concentrate, slowly at first and more and more quickly until it comes back to its initial value (up to a complex modulation) at a time around

$$\sqrt{\frac{\pi}{2}}\exp\frac{E_\gamma}{2} = \sqrt{\frac{\pi\lambda}{2\delta}}\exp\frac{\delta}{2\lambda}.$$

Indeed, in the proof of Proposition 2.2.9, we also proved implicitly that τ_γ is most of the time larger than 1 when E_γ is large. Even more, we proved that the time during which τ_γ is less than 1, given by

$$2\int_{\gamma_-}^1 \frac{\mathrm{d}x}{\sqrt{E_\gamma - \frac{1}{x^2} - 2\ln x}},$$

goes in fact to 0 as E_γ goes to ∞.

Since such very flat initial data give a solution which remains flat for a long time but then has a high "peak" which suddenly appears for a brief time, this behaviour might be related to rogue waves. For instance, the absolute value of the breather with $\delta = 35$ and $\lambda = 0.5$ in the initial data is plotted in Figure 2.2.3.

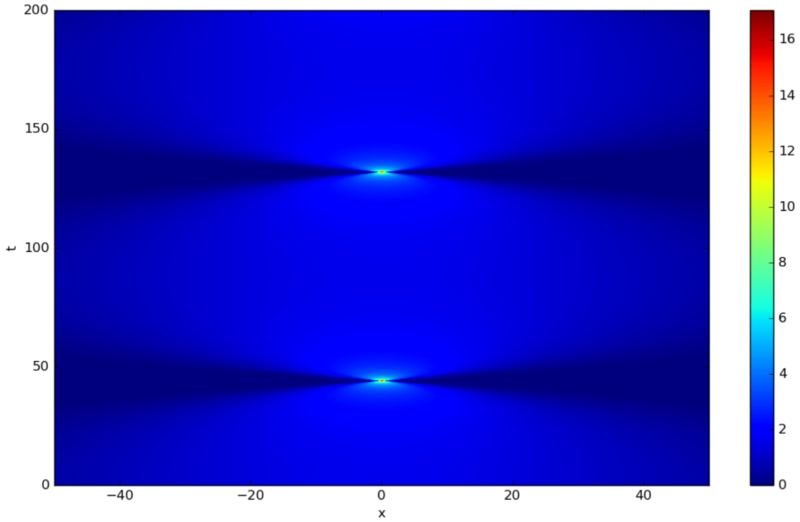

Figure 2.2.3 – Plot of the breather ($\lambda = 0.5$) with initial data $\exp\left(\frac{1}{2} - \delta x^2\right)$ with $\delta^{-1} = 2 \times 35^2$. The first "peak" is at $t = 43.86$.

2.2.4. Proof of Proposition 2.1.9

The proof of this proposition relies on the following result which characterizes $S_d(\mathbb{C})^{\mathrm{Re}\,++}$.

Lemma 2.2.10. *For any $A \in S_d(\mathbb{C})^{\mathrm{Re}\,++}$, there exists $R \in \mathcal{O}_d(\mathbb{R})$ and $\beta_1, \dots, \beta_d \in \mathbb{C}^+$ such that*

$$A = RDR^\top,$$

where

$$D = \begin{bmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_d \end{bmatrix}.$$

Proof. By definition, $A_r := \mathrm{Re}\,A$ and $A_i := \mathrm{Im}\,A$ are real symmetric and commute. Therefore, they can be orthogonally co-diagonalized, which means that we have a $R \in \mathcal{O}_d(\mathbb{R})$ and D_r and D_i real diagonal matrices such that $A_r = RD_rR^\top$ and $A_i = RD_iR^\top$. Moreover, since A_r is positive definite, the diagonal coefficients of D_r are positive. Therefore, $A = RDR^\top$ where $D = D_r + iD_i$ is a diagonal matrix whose diagonal coefficients are in $\mathbb{C}^+$. □

Proof of Proposition 2.1.9. Set $R \in \mathcal{O}_d(\mathbb{R})$, $\beta_1, \dots, \beta_d \in \mathbb{C}^+$ and D given by Lemma 2.2.10 for A. Since u^A is a solution to 2.1.1 which is invariant under an orthogonal transformation, $u^A(t, Rx)$ is also solution to 2.1.1 with initial data

$$u_{\mathrm{in}}^A(Rx) = \exp\left[\frac{d}{2} - x^\top Dx\right] = u_{\mathrm{in}}^{\alpha_1}(x_1)\dots u_{\mathrm{in}}^{\alpha_d}(x_d),$$

where for all $j \in \{1, \dots, d\}$

$$\alpha_j := \frac{1}{\sqrt{2\,\mathrm{Re}\,\beta_j}} - i\frac{\mathrm{Im}\,\beta}{\sqrt{2\,\mathrm{Re}\,\beta_j}}$$

Since every u^{α_j} is solution to 2.1.1 in dimension 1, we know that

$$u^{\alpha_1}(t) \otimes \dots \otimes u^{\alpha_d}(t)$$

is solution to 2.1.1 with the same previous initial data. Thus, by uniqueness of the solution in $\mathcal{C}_b(\mathbb{R}, W(\mathbb{R}^d))$, there holds

$$u^A(t, R\,.) = u^{\alpha_1}(t) \otimes \dots \otimes u^{\alpha_d}(t).$$

□

2.3. NONLINEAR SUPERPOSITION

In this section, we prove Theorem 2.1.10 (in any dimension $d \in \mathbb{N}^*$). This result is directly inspired from the energy estimate in L^2 found in [40] to prove the uniqueness of the solution for 2.1.1 in the case $\lambda < 0$. This energy estimate is the consequence of the following lemma:

Lemma 2.3.1 ([40, Lemma 1.1.1])**.** *There holds*

$$\left|\operatorname{Im}\Big((z_2 \ln|z_2|^2 - z_1 \ln|z_1|^2)(\overline{z_2} - \overline{z_1})\Big)\right| \leq 2|z_2 - z_1|^2, \qquad \forall z_1, z_2 \in \mathbb{C}.$$

Indeed, taking u_1 and u_2 two solutions to 2.1.1, $u \coloneqq u_1 - u_2$ satisfies

$$i\,\partial_t u + \frac{1}{2}\Delta u = -\lambda\Big(u_1 \ln|u_1|^2 - u_2 \ln|u_2|^2\Big).$$

Thus, we directly get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}^2 = -\lambda \operatorname{Im}\int \Big(u_1 \ln|u_1|^2 - u_2 \ln|u_2|^2\Big)(\overline{u_1} - \overline{u_2})\,\mathrm{d}x \leq 2|\lambda|\,\|u(t)\|_{L^2}^2.$$

We emphasize that this inequality does not involve the H^1 norm of u_1 or u_2: it only involves the L^2 norm of u. On the other hand, if v is solution to 2.1.1 with initial data $u_1(0) + u_2(0)$, one can wonder how close $v(t)$ will stay to $u(t) \coloneqq u_1(t) + u_2(t)$. If now we set $w \coloneqq v - (u_1 - u_2) = v - u$, then it satisfies

$$i\,\partial_t w + \frac{1}{2}\Delta w = -\lambda\Big(v \ln|v|^2 - u_1 \ln|u_1|^2 - u_2 \ln|u_2|^2\Big).$$

Therefore, there holds

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{L^2}^2 &= -\lambda \operatorname{Im}\int \Big(v\ln|v|^2 - u_1\ln|u_1|^2 - u_2\ln|u_2|^2\Big)(\overline{v} - \overline{u})\,\mathrm{d}x \\
&= -\lambda \operatorname{Im}\int \Big(v\ln|v|^2 - u\ln|u|^2\Big)(\overline{v} - \overline{u})\,\mathrm{d}x \\
&\quad -\lambda \operatorname{Im}\int \Big(u\ln|u|^2 - u_1\ln|u_1|^2 - u_2\ln|u_2|^2\Big)(\overline{v} - \overline{u})\,\mathrm{d}x \\
\frac{1}{2}\left|\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{L^2}^2\right| &\leq 2|\lambda|\,\|w\|_{L^2}^2 + |\lambda|\int \Big|u\ln|u|^2 - u_1\ln|u_1|^2 - u_2\ln|u_2|^2\Big||w|\,\mathrm{d}x \\
&\leq 2|\lambda|\,\|w\|_{L^2}^2 + |\lambda|\Big\|u\ln|u|^2 - u_1\ln|u_1|^2 - u_2\ln|u_2|^2\Big\|_{L^2}\|w\|_{L^2}.
\end{aligned}$$

Dividing by $\|w\|_{L^2}$, we obtain

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{L^2}\right| \leq 2|\lambda|\,\|w\|_{L^2} + |\lambda|\Big\|u\ln|u|^2 - u_1\ln|u_1|^2 - u_2\ln|u_2|^2\Big\|_{L^2}.$$

This estimate can also be generalized to more than 2 solutions with the same computation: for any integer $N \geq 2$ and any solutions $v, u_1, \ldots, u_N$ to 2.1.1, the function $w \coloneqq v - u$ where $u \coloneqq \sum u_j$ satisfies the inequality

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{L^2}\right| \leq 2|\lambda|\,\|w\|_{L^2} + |\lambda|\left\|u\ln|u|^2 - \sum_{j=1}^{N} u_j \ln|u_j|^2\right\|_{L^2}. \tag{2.3.1}$$

This estimate can be useful up to two conditions. First, we must know a time t_0 where $v(t_0)$ and $u(t_0)$ are close in L^2. Then, we also need a way to estimate the last term in the right-hand side. This term should be small for instance if the u_is are "well separated". Such a thing may be hard to prove in general, but it is easier if we have an explicit expression for the u_is. This is the case for the breathers and Gaussons, or more generally for the Gaussian functions solution. In particular, if their centers are far away from each other, then this term is actually very small:

Lemma 2.3.2. *For any $d \in \mathbb{N}^*$, there exists $C_d > 0$ such that the following holds. Let $N \in \mathbb{N}^*$ and take $x_k \in \mathbb{R}^d$, $\omega_k \in \mathbb{R}$, $\Lambda_k \in S_d(\mathbb{C})^{\mathrm{Re}+}$ and $\theta_k : \mathbb{R}^d \to \mathbb{R}$ a real measurable function for $k = 1, \ldots, N$, and define for all $x \in \mathbb{R}^d$*

$$g_k(x) = \exp\left[i\theta_k(x) + \omega_k - (x - x_k)^\top \Lambda_k (x - x_k)\right],$$

as well as

$$g(x)=\sum_{k=1,\dots,N} g_k(x).$$

If

$$\varepsilon := \Big(\min_{k\neq j}|x_j-x_k|\Big)^{-1} < \varepsilon_0 := \min\Bigg(\frac{\sqrt{\lambda_+}}{\max(\sqrt{\delta\omega+1},\sqrt{\ln N})},\sqrt{\frac{\lambda_-}{d+2}}\Bigg)$$

where $\delta\omega := \max_{j,k}|\omega_k-\omega_j|$, $\lambda_+ = \max_k \operatorname{Re}\sigma(\Lambda_k)$ *and* $\lambda_- = \min_k \operatorname{Re}\sigma(\Lambda_k) > 0$, *then*

$$\Bigg\|g\ln|g| - \sum_{k=1}^N g_k\ln|g_k|\Bigg\|_{L^2(\mathbb{R}^d)} \le C_d N^{\frac{3}{2}}\,\frac{\lambda_+}{\varepsilon^{\frac{d}{2}+1}\sqrt{\lambda_-}}\,\exp\left[-\frac{\lambda_-}{4\varepsilon^2}+\max_j\omega_j\right].$$

Such an estimate allows us to prove Theorem 2.1.10.

Proof of Theorem 2.1.10. Thanks to Proposition 2.1.5 and 2.1.5, we know that each $G_k := G^d_{A_k,\omega_k,x_k,v,\theta_k}$ can be written under the form

$$G_k(t,x) = \exp\left[i\theta_k(t,x)+\tilde\omega_k(t)-(x-x_k-vt)^\top A_k(t)(x-x_k-vt)\right],$$

with

$$\tilde\omega_k(t) = \omega_k + \frac{d}{2} - \frac{1}{4}\ln\frac{\det\operatorname{Re}A_k(t)}{\det\operatorname{Re}A_k(0)}.$$

In particular, we have

$$\sup_{t,\ell,k}|\tilde\omega_\ell(t)-\tilde\omega_k(t)| \le \delta\tilde\omega := \delta\omega + \frac{d}{2}\ln\frac{\tau_+}{\tau_-},$$

where $\tau_+ = \sup_{t,k}\operatorname{Re}\sigma(A_k(t))$ and $\tau_- = \inf_{t,k}\operatorname{Re}\sigma(A_k(t))$. Hence, setting $G := \sum_k G_k$ and

$$\varepsilon_0 := \frac{1}{\sqrt{2}}\min\Bigg(\frac{\sqrt{\tau_+}}{\max(\sqrt{\delta\tilde\omega+1},\sqrt{\ln N})},\sqrt{\frac{\tau_-}{\frac{d}{2}+1}}\Bigg),$$

and since $|x_k-vt-(x_j-vt)| = |x_k-x_j| \ge \varepsilon^{-1}$ for all $k\neq j$, there holds from Lemma 2.3.2

$$\Bigg\|G\ln|G| - \sum_{k=1}^N G_k\ln|G_k|\Bigg\|_{L^2(\mathbb{R}^d)} \le C_d N^{\frac{3}{2}}\,\frac{\tau_+}{\varepsilon^{\frac{d}{2}+1}\sqrt{\tau_-}}\,\exp\left[-\frac{\tau_-}{4\varepsilon^2}+\max_j\omega_j\right]$$

as soon as $\varepsilon < \varepsilon_0$. Plugging this into 2.3.1, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{L^2} \le 2|\lambda|\,\|w\|_{L^2} + C_d N^{\frac{3}{2}}\,\frac{\lambda\,\tau_+}{\varepsilon^{\frac{d}{2}+1}\sqrt{\tau_-}}\,\exp\left[-\frac{\tau_-}{4\varepsilon^2}+\max_j\omega_j\right],$$

where $w = u - G$. The result readily follows from the Gronwall lemma and the fact that $w(0)=0$. ☐

2.3.1. Proof of Lemma 2.3.2

This lemma shows that we can approximate the non-linearity in the equation for the sum of the Gaussian functions by the sum of the non-linearity of each Gaussian, as soon as these Gaussian are well separated. This kind of result is rather usual, as we often find it when talking about multi-solitons for instance. The proof usually uses the exponential decay at infinity of the solitons along with the fact that the non-linearity is locally Lipschitz. Here, we have a better decay at infinity for our functions (which are Gaussian), but our non-linearity $F(z) := z\ln|z|^2$ is not Lipschitz at 0, therefore this kind of result is not obvious at first sight. However, F is actually almost Lipschitz in the following sense:

Lemma 2.3.3. *For all* $z,\tilde z\in\mathbb{C}$ *such that* $|z|\le 1$, $|\tilde z|\le 1$ *and* $z\neq 0$, *there holds*

$$|F(\tilde z)-F(z)| \le |z-\tilde z|\Big[6-\ln|z|^2\Big].$$

This lemma is interesting: for any $x, \tilde{x} \in (0,1]$, the mean value inequality would only give

$$|F(x) - F(\tilde{x})| \leq |x - \tilde{x}|\Big[6 - 2\ln\min(x, \tilde{x})\Big].$$

In particular, if we fix x and make $\tilde{x}$ goes to 0, the right-hand side goes to ∞ whereas the left-hand side only goes to $F(x) = x\ln x^2$. Therefore the above inequality is not optimal and not fitted when we take $\tilde{x}$ which may be very small compared to x or even sometimes vanish. This lemma shows that we can actually take either $\ln x$ or $\ln\tilde{x}$ in the right-hand side without having to know which one of x or $\tilde{x}$ is the smallest. Another advantage is that the expression of any $\ln|g_j(x)|^2$ is clearly simpler than $\ln|g(x)|^2$, which will allow us to have simpler computations when applying the previous lemma with $z = g_j(x)$ and $\tilde{z} = g(x)$.

Proof. For this proof only, we use the identification $\mathbb{C} \approx \mathbb{R}^2$, and we see F as a function from $\mathbb{R}^2$ to itself :

$$F : z = \begin{bmatrix} z_r \\ z_i \end{bmatrix} \in \mathbb{R}^2 \mapsto z\ln|z|^2 = \begin{bmatrix} z_r\ln|z|^2 \\ z_i\ln|z|^2 \end{bmatrix}.$$

Then, F is differentiable on $\mathbb{C}\setminus\{0\}$ and we can compute for $z \neq 0$

$$DF(z) = \begin{bmatrix} \ln|z|^2 + 2\frac{z_r^2}{|z|^2} & 2\frac{z_r z_i}{|z|^2} \\ 2\frac{z_r z_i}{|z|^2} & \ln|z|^2 + 2\frac{z_i^2}{|z|^2} \end{bmatrix} = R_z^{-1}\begin{bmatrix} \ln|z|^2 + 2 & 0 \\ 0 & \ln|z|^2 \end{bmatrix} R_z,$$

where R_z is the rotation which maps z onto the real positive half-line. Hence, there holds for all $z \in \mathbb{C}^*$

$$\|DF(z)\|_2 \leq 2(|\ln|z|| + 1).$$

Then, we compute for z and $\tilde{z}$ satisfying the assumptions above:

$$\begin{aligned} F(\tilde{z}) - F(z) &= \int_0^1 DF(z + t(\tilde{z} - z))\,(\tilde{z} - z)\,\mathrm{d}t, \\ |F(\tilde{z}) - F(z)| &\leq \int_0^1 \|DF(z + t(\tilde{z} - z))\|_2\,\mathrm{d}t\,|\tilde{z} - z| \\ &\leq 2\int_0^1 \Big(\Big|\ln|z + t(\tilde{z} - z)|\Big| + 1\Big)\,\mathrm{d}t\,|\tilde{z} - z|. \end{aligned} \tag{2.3.2}$$

By assumption, there holds:

$$\Big||z| - t|\tilde{z} - z|\Big| \leq |z + t(\tilde{z} - z)| \leq (1-t)|z| + t|\tilde{z}| \leq 1.$$

Since $y \mapsto \ln y$ is increasing and non-positive on $(0,1]$, we have for a.e. $t \in [0,1]$

$$\Big|\ln|z + t(\tilde{z} - z)|\Big| = -\ln|z + t(\tilde{z} - z)| \leq -\ln\Big||z| - t|\tilde{z} - z|\Big|.$$

Putting this in 2.3.2, we get

$$\begin{aligned} |F(\tilde{z}) - F(z)| &\leq 2\int_0^1 (1 - \ln\Big||z| - t|\tilde{z} - z|\Big|)\,\mathrm{d}t\,|\tilde{z} - z| = 2\int_{|z| - |\tilde{z} - z|}^{|z|} (1 - \ln|v|)\,\mathrm{d}v \\ &\leq 2\Big[2v - v\ln|v|\Big]_{|z| - |\tilde{z} - z|}^{|z|} \\ &\leq 4|\tilde{z} - z| + 2(|z| - |\tilde{z} - z|)\ln\Big||z| - |\tilde{z} - z|\Big| - 2|z|\ln|z|. \end{aligned}$$

Then, we need to estimate the difference between the two last terms with the following lemma.

Lemma 2.3.4. *For any $a \in (0,1]$ and $\delta \geq 0$ such that $a - \delta \geq -1$, there holds*

$$(a - \delta)\ln|a - \delta| - a\ln a \leq \delta(1 - \ln a).$$

The conclusion follows from applying this lemma with $a = |z|$ and $\delta = |\tilde{z} - z|$. □

Proof of Lemma 2.3.4. Take a and δ satisfying the assumptions of the Lemma.

— If $\delta < a$, then $0 \leq a - \delta \leq a$, so in particular $\ln|a - \delta| \leq \ln a$, which yields

$$(a - \delta)\ln|a - \delta| - a\ln a \leq (a - \delta)\ln a - a\ln a = -\delta\ln a.$$

— If $\delta \geq 2a$, in the same way, we have $a - \delta \leq -a < 0$, in particular $\ln|a-\delta| \geq \ln a$ which yields

$$(a-\delta)\ln|a-\delta| - a\ln a \leq (a-\delta)\ln a - a\ln a = -\delta \ln a.$$

— Otherwise, if $a \leq \delta < 2a$, then $-1 < \frac{a-\delta}{a} = 1 - \frac{\delta}{a} \leq 0$ and we can compute

$$(a-\delta)\ln|a-\delta| - a\ln a = (a-\delta)\ln\left|\frac{a-\delta}{a}\right| - \delta\ln a \leq a - \delta \ln a \leq \delta(1-\ln a).$$

where we have used the fact that $y\ln|y| \leq 1$ for all $y \in [-1, 0]$. □

Substituting $\tilde{z}$ by $g(x)$ and z by $g_k(x)$ (which does not vanish) for some k, we see that the $\ln|z|^2$ in the right-hand side becomes a quadratic function in x, which is totally harmless compared with the decay at infinity of the Gaussons, provided that we use this inequality with g_k only where g_k is "predominant". To be more precise, we will apply such an estimate on I_k defined by

$$I_k = \left\{x \in \mathbb{R}^d, |x - x_k| \leq |x - x_j| \quad \forall j \neq k\right\}.$$

It is easy to see that, since the x_i are all different from each other, there holds $\mathcal{L}_d(I_j \cap I_k) = 0$ for all $j \neq k$ where $\mathcal{L}_d$ is the Lebesgue measure in $\mathbb{R}^d$. Moreover, there also holds $\mathbb{R}^d = \bigcup_j I_j$, so that

$$\int_{\mathbb{R}^d} = \sum_j \int_{I_j}.$$

However, we also need $|g(x)|$ and $|g_k(x)|$ to be smaller than 1 everywhere in order to apply the previous lemma, which means that we need the ω_js to be negatively large enough, but we show that we can stick to this case.

Proposition 2.3.5. *Set $\omega := \max_j \omega_j$. For all $x \in \mathbb{R}^d$, there holds*

$$\sum_j |g_j(x)| \leq N e^{\omega}.$$

Proof. It easily follows from the fact that for any j and any $x \in \mathbb{R}^d$, there holds $|g_j(x)| \leq e^{\omega_j} \leq e^{\omega}$ □

We define $\tilde{g}_k := N^{-1}e^{-\omega}\, g_k$ and $\tilde{g} := N^{-1}e^{-\omega}\, g$, so that

$$|\tilde{g}(x)| \leq \sum_k |\tilde{g}_k(x)| \leq 1,$$

$$\left| g(x)\ln|g(x)|^2 - \sum_{j=1}^N g_j(x)\ln|g_j(x)|^2 \right| = N e^{\omega} \left| \tilde{g}(x)\ln|\tilde{g}(x)|^2 - \sum_{j=1}^N \tilde{g}_j(x)\ln|\tilde{g}_j(x)|^2 \right|. \tag{2.3.3}$$

Then, we can use Lemma 2.3.3, which leads to:

Proposition 2.3.6. *For all $k \in \{1, \dots, N\}$ and $x \in \mathbb{R}^d$, there holds*

$$\left| \tilde{g}(x)\ln|\tilde{g}(x)|^2 - \sum_{j=1}^N \tilde{g}_j(x)\ln|\tilde{g}_j(x)|^2 \right| \\ \leq 2\sum_{j\neq k} |\tilde{g}_j(x)| \Big[\delta\omega_j + \delta\omega_k + 3 + 2\ln N + \lambda_+|x-x_k|^2 + \lambda_+|x-x_j|^2\Big],$$

where $\delta\omega_j := \omega - \omega_j$ for all j.

Proof. We recall that for all j,

$$|\tilde{g}_j(x)| = \exp\Big[-\delta\omega_j - \ln N - (x-x_j)^\top \Lambda_j (x - x_j)\Big],$$

which also yields that

$$-\ln|\tilde{g}_j(x)| \leq \delta\omega_j + \ln N + \lambda_+|x-x_j|^2.$$

Then, we can easily compute

$$\begin{aligned}
&\left|\tilde{g}(x)\ln|\tilde{g}(x)| - \sum_{j=1}^{N}\tilde{g}_j(x)\ln|\tilde{g}_j(x)|\right| \\
\leq &\Big|\tilde{g}(x)\ln|\tilde{g}(x)| - \tilde{g}_k(x)\ln|\tilde{g}_k(x)|\Big| + \left|\sum_{j\neq k}\tilde{g}_j(x)\ln|\tilde{g}_j(x)|\right| \\
\leq &|\tilde{g}(x)-\tilde{g}_k(x)|\Big[3-\ln|\tilde{g}_k(x)|\Big] + \sum_{j\neq k}|\tilde{g}_j(x)||\ln|\tilde{g}_j(x)|| \\
\leq &\left|\sum_{j\neq k}\tilde{g}_j(x)\right|\Big[3-\ln|\tilde{g}_k(x)|\Big] + \sum_{j\neq k}|\tilde{g}_j(x)||\ln|\tilde{g}_j(x)|| \\
\leq &\sum_{j\neq k}|\tilde{g}_j(x)|\Big[3-\ln|\tilde{g}_k(x)|-\ln|\tilde{g}_j(x)|\Big].
\end{aligned}$$

The conclusion follows. □

Thanks to 2.3.3, we can readily come back in terms of g and g_j.

Corollary 2.3.7. *For all $k \in \{1,\dots,N\}$ and $x \in \mathbb{R}^d$, there holds*

$$\begin{aligned}
&\left|g(x)\ln|g(x)|^2 - \sum_{j=1}^{N} g_j(x)\ln|g_j(x)|^2\right| \\
&\quad \leq 2\sum_{j\neq k}|g_j(x)|\Big[\delta\omega_j + \delta\omega_k + 3 + 2\ln N + \lambda_+|x-x_k|^2 + \lambda_+|x-x_j|^2\Big].
\end{aligned} \tag{2.3.4}$$

Thus, we can estimate this difference in $L^2(I_k)$ norm.

Proposition 2.3.8. *If $\varepsilon \leq \varepsilon_0$ (where ε_0 is defined in Lemma 2.3.2) then*

$$\left\|g(x)\ln|g(x)|^2 - \sum_{j=1}^{N} g_j(x)\ln|g_j(x)|^2\right\|_{L^2(I_k)} \leq C_d N\,\frac{\lambda_+}{\varepsilon^{\frac{d}{2}+1}\sqrt{\lambda_-}}e^{\omega-\frac{\lambda_-}{8\varepsilon^2}}.$$

To prove this result, we need to use Corollary 2.3.7 and estimate the $L^2(I_k)$ norm of each term of the sum in 2.3.4. For this, we will use the following lemma:

Lemma 2.3.9. *For any $d \in \mathbb{N}^*$, there exists $C_d > 0$ such that for all $\gamma > 0$, $R > 0$ and $x_0 \in \mathbb{R}^d$ such that $R \geq \gamma^{-\frac{1}{2}}$ and $|x_0| \leq 2R$, there holds*

$$\begin{aligned}
\int_{\mathbb{R}^d\setminus B(0,R)} |x|^4 e^{-\gamma|x|^2}\,\mathrm{d}x &\leq C_d\frac{R^{d+2}}{\gamma}e^{-\gamma R^2}, \\
\int_{\mathbb{R}^d\setminus B(0,R)} e^{-\gamma|x|^2}\,\mathrm{d}x &\leq C_d\frac{R^{d-2}}{\gamma}e^{-\gamma R^2}, \\
\int_{\mathbb{R}^d\setminus B(0,R)} |x-x_0|^4 e^{-\gamma|x|^2}\,\mathrm{d}x &\leq C_d\frac{R^{d+2}}{\gamma}e^{-\gamma R^2}.
\end{aligned}$$

Before proving this Lemma, we recall the usual and useful estimate for the Gauss error function.

Lemma 2.3.10. *For any $y \geq 1$ and $\gamma > 0$, there holds*

$$\int_y^\infty e^{-\gamma x^2}\,\mathrm{d}x < \frac{1}{2\gamma y}e^{-\gamma y^2}.$$

Proof of Lemma 2.3.10. We easily compute:

$$\int_y^\infty e^{-\gamma x^2}\,\mathrm{d}x < \int_y^\infty \frac{x}{y}e^{-\gamma x^2}\,\mathrm{d}x = \frac{1}{y}\left[-\frac{e^{-\gamma x^2}}{2\gamma}\right]_y^\infty = \frac{1}{2\gamma y}e^{-\gamma y^2}.$$ □

Proof of Lemma 2.3.9. For the first estimate, a radial change of variables yields

$$\int_{\mathbb{R}^d\setminus B(0,R)} |x|^4 e^{-\gamma|x|^2}\,\mathrm{d}x = C_d \int_R^\infty r^{3+d} e^{-\gamma r^2}\,\mathrm{d}r = C_d \times J_{3+d},$$

where $J_m = \int_R^\infty r^m e^{-\gamma r^2}\,\mathrm{d}r$ for any $m \in \mathbb{N}$. With an integration by parts, we get

$$J_{m+2} = \frac{1}{2\gamma} R^{m+1} e^{-\gamma R^2} + \frac{m+1}{2\gamma} J_m.$$

Since we have $R > \gamma^{-\frac{1}{2}}$ and since there holds

$$J_0 < \frac{1}{2\gamma R} e^{-\gamma R^2} \qquad \text{and} \qquad J_1 = \frac{1}{2\gamma} e^{-\gamma R^2},$$

we can easily prove by induction that

$$J_m \le \frac{C_m}{\gamma} R^{m-1} e^{-\gamma R^2},$$

which leads to the first estimate. The second estimate can be proved in the same way. As for the third estimate, we use the fact that for $x \in \mathbb{R}^d$

$$|x - x_0|^4 \le C_0(|x|^4 + |x_0|^4),$$

which yields

$$\begin{aligned}
&\int_{\mathbb{R}^d\setminus B(0,R)} |x-x_0|^4 e^{-\gamma|x|^2}\,\mathrm{d}x \\
\le& C_0\left(\int_{\mathbb{R}^d\setminus B(0,R)} |x|^4 e^{-\gamma|x|^2}\,\mathrm{d}x + |x_0|^4 \int_{\mathbb{R}^d\setminus B(0,R)} e^{-\gamma|x|^2}\,\mathrm{d}x\right) \\
\le& C_d\left(\int_R^\infty r^{3+d} e^{-\gamma r^2}\,\mathrm{d}r + |x_0|^4 \int_R^\infty r^{d-1} e^{-\gamma r^2}\,\mathrm{d}r\right) \\
\le& C_d\left(\frac{R^{d+2}}{\gamma} e^{-\gamma R^2} + \frac{|x_0|^4 R^{d-2}}{\gamma} e^{-\gamma R^2}\right) \\
\le& C_d \frac{R^{d+2}}{\gamma} e^{-\gamma R^2}.
\end{aligned}$$

□

Proof of Proposition 2.3.8. Thanks to Corollary 2.3.7, we have

$$\begin{aligned}
&\left\| g(x)\ln|g(x)| - \sum_{j=1}^N g_j(x)\ln|g_j(x)| \right\|_{L^2(I_k)} \\
\le& \sum_{j\neq k} \|g_j(x)\|_{L^2(I_k)} \Big[\delta\omega_j + \delta\omega_k + 3 + 2\ln N\Big] \\
&\quad + \lambda_+ \left\| g_j(x)|x-x_k|^2 \right\|_{L^2(I_k)} + \lambda_+ \left\| g_j(x)|x-x_j|^2 \right\|_{L^2(I_k)}.
\end{aligned} \tag{2.3.4}$$

For $j \neq k$, we know that for any $x \in I_k$, there holds

$$|x_j - x_k| \le |x - x_j| + |x - x_k| \le 2|x - x_j|,$$

and thus

$$|x - x_j| \ge \frac{1}{2}|x_j - x_k|.$$

Hence, $I_k \subset \mathbb{R}^d \setminus B(x_j, R_j^k)$ where $R_j^k := \frac{1}{2}|x_j - x_k|$. Therefore, using also the fact that $y^\top \Lambda_j y \ge \lambda_- |y|^2$ for all $y \in \mathbb{R}^d$, we get

$$\begin{aligned}
\|g_j(x)\|^2_{L^2(I_k)} &\le \int_{\mathbb{R}^d\setminus B(x_j,R_j^k)} \exp\Big[2\omega_j - 2\lambda_-|x-x_j|^2\Big], \\
\left\| g_j(x)|x-x_k|^2 \right\|^2_{L^2(I_k)} &\le \int_{\mathbb{R}^d\setminus B(x_j,R_j^k)} |x-x_k|^4 \exp\Big[2\omega_j - 2\lambda_-|x-x_j|^2\Big],
\end{aligned}$$

$$\Big\|g_j(x)|x-x_j|^2\Big\|^2_{L^2(I_k)} \leq \int_{\mathbb{R}^d\setminus B(x_j,R_j^k)} |x-x_j|^4 \exp\Big[2\omega_j - 2\lambda_-|x-x_j|^2\Big].$$

Since $\omega_j \leq \omega$, and with a change of variable, we get

$$\|g_j(x)\|^2_{L^2(I_k)} \leq \int_{\mathbb{R}^d\setminus B(0,R_j^k)} \exp\Big[2\omega - 2\lambda_-|y|^2\Big],$$

$$\Big\|g_j(x)|x-x_k|^2\Big\|^2_{L^2(I_k)} \leq \int_{\mathbb{R}^d\setminus B(0,R_j^k)} |y-(x_k-x_j)|^4 \exp\Big[2\omega - 2\lambda_-|y|^2\Big],$$

$$\Big\|g_j(x)|x-x_j|^2\Big\|^2_{L^2(I_k)} \leq \int_{\mathbb{R}^d\setminus B(0,R_j^k)} |y|^4 \exp\Big[2\omega - 2\lambda_-|y|^2\Big].$$

Now, we apply Lemma 2.3.9 in order to estimate all the $L^2(I_k)$ norms of the right-hand side. However, we need to check the assumptions. We already know that $|x_k - x_j| = 2R_j^k$. Moreover, there holds from the same equality and with the definition of ε

$$R_j^k \geq \frac{1}{2\varepsilon}. \tag{2.3.5}$$

The fact that $\varepsilon_0 \leq \sqrt{\frac{\lambda_-}{2}}$ yields $R_j^k \geq (2\lambda_-)^{-\frac{1}{2}}$. Along with the fact that $\omega_j \leq \omega$, this leads to

$$\|g_j(x)\|^2_{L^2(I_k)} \leq C_d \frac{(R_j^k)^{d-2}}{\lambda_-} e^{2\omega - 2\lambda_-(R_j^k)^2},$$

$$\Big\|g_j(x)|x-x_k|^2\Big\|^2_{L^2(I_k)} \leq C_d \frac{(R_j^k)^{d+2}}{\lambda_-} e^{2\omega - 2\lambda_-(R_j^k)^2},$$

$$\Big\|g_j(x)|x-x_j|^2\Big\|^2_{L^2(I_k)} \leq C_d \frac{(R_j^k)^{d+2}}{\lambda_-} e^{2\omega - 2\lambda_-(R_j^k)^2}.$$

Since we also assumed

$$\varepsilon_0 \leq \frac{\sqrt{\lambda_+}}{\max(\sqrt{\delta\omega+1}, \sqrt{\ln N})},$$

2.3.5 leads to

$$\max(\delta\omega+1, \ln N) \leq \frac{\lambda_+}{\varepsilon^2} \leq 4\lambda_+(R_j^k)^2,$$

so that

$$\Big[\delta\omega_j + \delta\omega_k + 3 + 2\ln N\Big]\|g_j(x)\|_{L^2(I_k)} \leq C_d\, \lambda_+ \frac{(R_j^k)^{\frac{d}{2}+1}}{\sqrt{\lambda_-}} e^{\omega - \lambda_-(R_j^k)^2},$$

$$\lambda_+\Big\|g_j(x)|x-x_k|^2\Big\|_{L^2(I_k)} \leq C_d\, \lambda_+ \frac{(R_j^k)^{\frac{d}{2}+1}}{\sqrt{\lambda_-}} e^{\omega - \lambda_-(R_j^k)^2},$$

$$\lambda_+\Big\|g_j(x)|x-x_j|^2\Big\|_{L^2(I_k)} \leq C_d\, \lambda_+ \frac{(R_j^k)^{\frac{d}{2}+1}}{\sqrt{\lambda_-}} e^{\omega - \lambda_-(R_j^k)^2}.$$

Moreover, we know that the function $f_d(\xi) := \xi^{\frac{d}{2}+1} e^{-\lambda_-\xi^2}$ is decreasing for $\xi \in \left[\sqrt{\frac{\frac{d}{2}+1}{2\lambda_-}}, \infty\right)$. Therefore, as there also holds $\varepsilon_0 \leq \sqrt{\frac{\lambda_-}{d+2}}$, then 2.3.5 also yields

$$R_j^k \geq \frac{1}{2\varepsilon} \geq \sqrt{\frac{\frac{d}{2}+1}{2\lambda_-}},$$

hence

$$\Big[\delta\omega_j + \delta\omega_k + 3 + 2\ln N\Big]\|g_j(x)\|_{L^2(I_k)} \leq C_d\, \frac{\lambda_+}{\varepsilon^{\frac{d}{2}+1}\sqrt{\lambda_-}} e^{\omega - \frac{\lambda_-}{4\varepsilon^2}},$$

$$\lambda_+\Big\|g_j(x)|x-x_k|^2\Big\|_{L^2(I_k)} \leq C_d\, \frac{\lambda_+}{\varepsilon^{\frac{d}{2}+1}\sqrt{\lambda_-}} e^{\omega - \frac{\lambda_-}{4\varepsilon^2}},$$

$$\lambda_+\Big\|g_j(x)|x-x_j|^2\Big\|_{L^2(I_k)} \leq C_d\, \frac{\lambda_+}{\varepsilon^{\frac{d}{2}+1}\sqrt{\lambda_-}} e^{\omega - \frac{\lambda_-}{4\varepsilon^2}}.$$

We get the result by putting these into 2.3.4. $\square$

Proof of Lemma 2.3.2. Define ε_0 as in Proposition 2.3.8. By definition of the sets I_k ($k \in \{1, \dots, N\}$) and using Proposition 2.3.8 for $\varepsilon \leq \varepsilon_0$, we get

$$\begin{aligned}\left\| g \ln|g| - \sum_{j=1}^{N} g_j \ln|g_j| \right\|_{L^2(\mathbb{R}^d)}^2 &= \sum_k \left\| g \ln|g| - \sum_{j=1}^{N} g_j \ln|g_j| \right\|_{L^2(I_k)}^2 \\ &\leq \sum_k C_d N^2 \frac{(\lambda_+)^2}{\varepsilon^{d+2}\lambda_-} e^{2\omega - \frac{\lambda_-}{2\varepsilon^2}} \\ &\leq C_d N^3 \frac{(\lambda_+)^2}{\varepsilon^{d+2}\lambda_-} e^{2\omega - \frac{\lambda_-}{2\varepsilon^2}}.\end{aligned}$$

□

Chapitre 3

Existence of multi-solitons for the focusing Logarithmic Non-Linear Schrödinger Equation

Abstract. We consider the logarithmic Schrödinger equation (logNLS) in the focusing regime. For this equation, Gaussian initial data remains Gaussian. In particular, the Gausson - a time-independent Gaussian function - is an orbitally stable solution. In this paper, we construct *multi-solitons* (or *multi-Gaussons*) for logNLS, with estimates in $H^1 \cap \mathcal{F}(H^1)$. We also construct solutions to logNLS behaving (in L^2) like a sum of N Gaussian solutions with different speeds (which we call *multi-gaussian*). In both cases, the convergence (as $t \to \infty$) is faster than exponential. We also prove a rigidity result on these constructed multi-gaussians and multi-solitons, showing that they are the only ones with such a convergence.

Résumé. On considère l'équation de Schrödinger logarithmique (logNLS) en régime focalisant. Pour cette équation, les données initiales gaussiennes restent gaussiennes. En particulier, le Gausson - une fonction gaussienne indépendante du temps - est une solution orbitalement stable. Dans cet article, nous construisons des *multi-solitons* (ou *multi-Gaussons*) pour logNLS, avec estimées dans $H^1 \cap \mathcal{F}(H^1)$. Nous construisons également des solutions à logNLS se comportant (dans L^2) comme une somme de N solutions gaussiennes avec différentes vitesses (que nous appelons *multi-gaussiennes*). Pour chaque cas, la convergence (pour $t \to \infty$) est plus rapide qu'exponentielle. Nous prouvons également un résultat de rigidité sur ces multi-gaussiennes et multi-solitons construits, en montrant que ce sont les seuls avec une telle convergence.

3.1. Introduction

3.1.1. Setting

We are interested in the *Logarithmic Non-Linear Schrödinger Equation*

$$i\,\partial_t u + \frac{1}{2}\Delta u + \lambda u \ln |u|^2 = 0, \tag{3.1.1}$$

with $x \in \mathbb{R}^d$, $d \geq 1$, $\lambda \in \mathbb{R} \setminus \{0\}$. This equation was introduced as a model of nonlinear wave mechanics and in nonlinear optics ([23], see also [25, 95, 98, 102, 59]). The case $\lambda < 0$ (whose study of the Cauchy problem goes back to [40, 86]) was recently studied by R. Carles and I. Gallagher who made explicit an unusually faster dispersion with a universal behaviour of the modulus of the solution (see [33]). The knowledge of this behaviour was very recently improved with a convergence rate but also extended through the semiclassical limit in [67]. On the other hand, the case $\lambda > 0$ seems to be the more interesting from a physical point of view and has been studied formally and rigorously (see for instance [58, 98]). In particular, the existence and uniqueness of solutions to the Cauchy problem have been solved in [40]. Moreover, it has been proved to be the non dispersive case and also that the so called *Gausson*

$$G^d(x) := \exp\Big(\frac{d}{2} - \lambda |x|^2\Big), \qquad x \in \mathbb{R}^d, \tag{3.1.2}$$

and its derivates through the invariants of the equation (translation in space, Galilean invariance, multiplication by a complex constant) are explicit solutions to (3.1.1) and bound states for the energy functional. Several results address the orbital stability of the Gausson as well as the existence of other stationary solutions and Gaussian solutions to (3.1.1); see *e.g.* [23, 38, 58, 8]. In this article, we address the question of the existence of multi-solitons (*i.e.* multi-Gaussons), but also the existence of multi-gaussians.

Remark 3.1.1 (Effect of scaling factors). As noticed in [33], unlike what happens in the case of an homogeneous nonlinearity (classically of the form $|u|^p u$), replacing u with κu ($\kappa > 0$) in (3.1.1) has only little effect, since we have

$$i\,\partial_t(\kappa u) + \frac{1}{2}\Delta(\kappa u) + \lambda(\kappa u) \ln |\kappa u|^2 - 2\lambda(\ln \kappa)\kappa u = 0.$$

The scaling factor thus corresponds to a purely time-dependent gauge transform:

$$\kappa u(t,x)\, e^{-2it\lambda \ln \kappa}$$

solves (3.1.1). In particular, the L^2-norm of the initial datum does not influence the dynamics of the solution.

3.1.2. The Logarithmic Non-Linear Schrödinger Equation

The Logarithmic Non-Linear Schrödinger Equation was introduced by I. Białynicki-Birula and J. Mycielski ([23]) who proved that it is the only nonlinear Schrödinger theory in which *the separability of noninteracting systems* hold: for noninteracting subsystems, no correlations are introduced by the nonlinear term. Therefore, for any initial data of the form $u_{\text{in}} = u_{\text{in}}^1 \otimes u_{\text{in}}^2$, *i.e.*

$$u_{\text{in}}(x) = u_{\text{in}}^1(x_1)\, u_{\text{in}}^2(x_2), \qquad \forall x_1 \in \mathbb{R}^{d_1}, \forall x_2 \in \mathbb{R}^{d_2}, x = (x_1, x_2),$$

the solution u to (3.1.1) (in dimension $d = d_1 + d_2$) with initial data $u_{|t=0} = u_{\text{in}}$ is

$$u(t) = u^1(t) \otimes u^2(t)$$

where u^j is the solution to (3.1.1) in dimension d_j with initial data u_{in}^j $(j = 1, 2)$.

They also emphasized that the case $\lambda > 0$ is probably the most physically relevant. For this case, the Cauchy problem has already been studied in [40] (see also [39]). We define the energy space

$$W(\mathbb{R}^d) := \left\{ v \in H^1(\mathbb{R}^d), |v|^2 \ln |v|^2 \in L^1(\mathbb{R}^d) \right\},$$

which is a reflexive Banach space when endowed with a Luxembourg type norm (see [38]). We can also define the mass, the angular momentum and the energy for all $v \in W(\mathbb{R}^d)$:

$$M(v) := \|v\|_{L^2}^2, \qquad \mathcal{J}(v) := \operatorname{Im} \int_{\mathbb{R}^d} \overline{v}\, \nabla v \, \mathrm{d}x, \qquad E(v) := \frac{1}{2} \|\nabla v\|_{L^2}^2 - \lambda \int_{\mathbb{R}^d} |v|^2 (\ln |v|^2 - 1) \, \mathrm{d}x.$$

Theorem 3.1.2 ([40, Théorème 2.1], see also [39, Theorem 9.3.4])**.** *For $\lambda > 0$, for any initial data $u_{\text{in}} \in W(\mathbb{R}^d)$, there exists a unique, global solution $u \in \mathcal{C}_b(\mathbb{R}, W(\mathbb{R}^d))$. Moreover the mass $M(u(t))$, the angular momentum $\mathcal{J}(u(t))$ and the energy $E(u(t))$ are independent of time.*

It is also worth noticing that there is an energy estimate at the level L^2:

Lemma 3.1.3 ([40, Lemme 2.2.1])**.** *For $\lambda > 0$, for any solutions u and v to* (3.1.1) *given by Theorem 3.1.2 with initial data $u_{\text{in}}, v_{\text{in}} \in W(\mathbb{R}^d)$ respectively, there holds for all $s, t \in \mathbb{R}$,*

$$\|u(s) - v(s)\|_{L^2} \le e^{2\lambda |t-s|} \|u(t) - v(t)\|_{L^2}.$$

Another surprising feature of (3.1.1) is that any Gaussian data remains Gaussian ([23]).

Proposition 3.1.4. *Any Gaussian initial data*

$$\exp\Big[\frac{d}{2} - x^\top A^{\text{in}} x\Big],$$

with $A^{\text{in}} \in S_d(\mathbb{C})^{\mathrm{Re}+} := \left\{ M \in M_d(\mathbb{C}), M^\top = M, \operatorname{Re} M \in S_d(\mathbb{R})^{++} \right\}$ (where $^\top$ designates the transposition and $S_d(\mathbb{R})^{++}$ the space of real positive defined symmetric matrices), gives rise to a Gaussian solution $B^{A^{\text{in}}}$ to (3.1.1) *of the form*

$$B^{A^{\text{in}}}(t,x) := \left(\frac{\det \operatorname{Re} A(t)}{\det \operatorname{Re} A^{\text{in}}} \right)^{\frac{1}{4}} \exp\Big[\frac{d}{2} - i\,\Phi(t) - \frac{1}{2} x^\top A(t) x\Big], \tag{3.1.3}$$

where A and ϕ satisfy

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -iA(t)^2 + 2i\lambda \operatorname{Re} A(t), \qquad A(0) = A^{\text{in}}, \tag{3.1.4}$$

$$\Phi(t) := \frac{1}{2} \int_0^t \operatorname{Tr}(\operatorname{Re} A(s)) \, \mathrm{d}s - \frac{\lambda}{2} \int_0^t \ln\left(\frac{\det \operatorname{Re} A(s)}{\det \operatorname{Re} A^{\text{in}}} \right) \mathrm{d}s - d\lambda t.$$

Moreover, if $\lambda > 0$,

$$0 < \inf_t \sigma(\operatorname{Re} A(t)) \le \sup_t \sigma(\operatorname{Re} A(t)) < +\infty.$$

In parallel, we define their derivates through the invariants and the scaling effect:

$$B^{A^{\text{in}}}_{\omega,x_0,v,\theta}(t,x) := \exp\left[i\left(\theta + 2\lambda\omega t - v\cdot x + \frac{|v|^2}{2}t\right) + \omega\right] B^{A^{\text{in}}}(t, x - x_0 - vt), \qquad t\in\mathbb{R},\, x\in\mathbb{R}^d, \tag{3.1.5}$$

For such data, the evolution of the solution is given by a single matrix ODE, which can even be simplified in dimension 1 (see [33, 12, 69]):

Proposition 3.1.5. *For any $\alpha\in\mathbb{C}^+ := \{z\in\mathbb{C}, \operatorname{Re} z > 0\}$, consider the ordinary differential equation*

$$\ddot r_\alpha = \frac{1}{r_\alpha^3} - \frac{2\lambda}{r_\alpha}, \qquad r_\alpha(0) = \operatorname{Re}\alpha =: \alpha_r, \qquad \dot r_\alpha(0) = \operatorname{Im}\alpha =: \alpha_i.$$

It has a unique solution $r_\alpha\in\mathcal{C}^\infty(\mathbb{R})$ with values in $(0,\infty)$. Then, set

$$u^\alpha(t,x) := \sqrt{\frac{\alpha_r}{r_\alpha(t)}}\exp\Big[\frac12 - i\Phi(t) - \frac{x^2}{2r_\alpha(t)^2} + i\frac{\dot r_\alpha(t)}{r_\alpha(t)}\frac{x^2}{2}\Big], \qquad t,x\in\mathbb{R}, \tag{3.1.6}$$

where

$$\Phi(t) := \frac12\int_0^t \frac{1}{r_\alpha(s)^2}\,\mathrm{d}s + \lambda\int_0^t \ln\frac{r_\alpha(s)}{\alpha_r}\,\mathrm{d}s - \lambda t.$$

Then u^α is solution to (3.1.1) *in dimension $d=1$.*

Note that whichever the sign of λ, the energy E has no definite sign. The distinction between focusing or defocusing nonlinearity is thus a priori ambiguous. However, in the previous case of Gaussian data in dimension 1, the behaviour of r_α (and then that of u^α) has been proven to be sensibly different ([23, 33, 69]).

Proposition 3.1.6. *If $\lambda > 0$, then r_α is periodic. On the other hand, if $\lambda < 0$, then*

$$r_\alpha(t) \underset{t\to\infty}{\sim} 2t\sqrt{|\lambda|\ln t}.$$

For $\lambda > 0$, such solutions u^α are almost periodic in time (up to a time-depending complex argument), which motivates to call them (and their derivates through the invariants and the scaling effect) *breathers*. If those solutions are in dimension 1, they can be tensorized in order to find other solutions (also called *breathers*) in higher dimension, even though they may be not periodic in general (see [69]). However, in higher dimension $d\geq 2$, in the general case, the solutions (3.1.3) are not periodic (and not "almost" periodic) and cannot be put under the form of a tensorization of breathers (3.1.6) in dimension 1 (already noticed in [23]). Therefore, all the functions (3.1.5) will be called *(general) Gaussian solutions* to (3.1.1) in the rest of the article, even though breathers are obviously a particular case of general Gaussian solutions.

Moreover, the ambiguity about the focusing or defocusing case has been removed in the general case by [38] (case $\lambda > 0$) and [33] (case $\lambda < 0$). Indeed, in the latter, the authors show that all the solutions disperse in an unusually faster way (in the same way as for the Gaussian case) with a universal dynamic: after rescaling, the modulus of the solution converges to a universal Gaussian profile. On the other hand, it has been proved that $\lambda > 0$ is the focusing case because there is no dispersion for large times thanks to the following result.

Lemma 3.1.7 ([38, Lemma 3.3])**.** *Let $\lambda > 0$. For any $k < \infty$ such that*

$$L_k := \big\{v\in W(\mathbb{R}^d), \|v\|_{L^2} = 1, E(v)\leq k\big\} \neq \emptyset,$$

there holds

$$\inf_{\substack{v\in L_k\\ 1\leq p\leq\infty}} \|v\|_{L^p} > 0.$$

This lemma, along with the conservation of the energy and the invariance through scaling factors (with Remark 3.1.1), indicates that the solution to (3.1.1) is not dispersive, no matter how small the initial data are. For instance, its L^∞ norm is bounded from below: to be more precise, there holds for all $t\in\mathbb{R}$ (see the proof of the above result)

$$\|u(t)\|_{L^\infty} \geq \exp\Big[1 - \frac{E(u(t))}{2\lambda\, M(u(t))}\Big] = \exp\Big[1 - \frac{E(u_{\text{in}})}{2\lambda\, M(u_{\text{in}})}\Big].$$

Actually, a specific Gaussian function (3.1.2) called *Gausson* and its derivates through the invariants of the equation and the scaling effect,

$$G^d_{\omega,x_0,v,\theta}(t,x) := \exp\left[i\left(\theta + 2\lambda\omega t - v\cdot x + \frac{|v|^2}{2}t\right) + \frac{d}{2} + \omega - \lambda|x - x_0 - vt|^2\right], \qquad t\in\mathbb{R},\, x\in\mathbb{R}^d,$$

for any $\omega, \theta \in \mathbb{R}$, $x_0, v \in \mathbb{R}^d$, are known to be solutions to (3.1.1) for $\lambda > 0$, as proved in [58] (and already noticed in [23]). It has also been proved that other radial stationary solutions to (3.1.1) exist (see [20, 18, 58]), but the Gausson is clearly special since it is the unique positive $\mathcal{C}^2$ stationary solution to (3.1.1) (also proved in [58, 142]) and also since it is orbitally stable ([8], following the work of [38]).

Theorem 3.1.8 ([8, Theorem 1.5])**.** *Let $\omega \in \mathbb{R}$. For any $\varepsilon > 0$, there exists $\eta > 0$ such that for all $u_0 \in W(\mathbb{R}^d)$ satisfying*

$$\inf_{\theta, x_0} \left\| u_0 - e^{\omega + i\theta} G^d(. - x_0) \right\|_{W(\mathbb{R}^d)} < \eta,$$

the solution $u(t)$ of (3.1.1) *with initial data u_0 satisfies*

$$\sup_t \inf_{\theta, x_0} \left\| u(t) - e^{\omega + i\theta} G^d(. - x_0) \right\|_{W(\mathbb{R}^d)} < \varepsilon.$$

Remark 3.1.9. Remark that

$$G^d_{\omega, x_0, v, \theta} \equiv B^{2\lambda I_d}_{\omega, x_0, v, \theta},$$

where I_d is the identity matrix in dimension d. Indeed, $2\lambda I_d$ is a constant matrix solution to (3.1.4).

Remark 3.1.10. In particular, we point out that solitary wave solutions for (3.1.1) (*i.e.* Gaussons) exist for ALL frequencies, unlike NLS equations with polynomial-like nonlinearity for which the only possible frequencies are (at least) non-negative. This is a consequence of the logarithmic nonlinearity, which satisfies

$$g(s) \coloneqq -\lambda \ln s \underset{s \to 0}{\longrightarrow} +\infty,$$

unlike polynomial-like nonlinearity.

3.1.3. Main results

Existence of multi-Gaussons. It was observed and proved for the Korteweg-de Vries equation that, for a large class of initial data, all solutions are global and eventually decompose into a finite sum of solitons going to the right and a dispersive part going to the left [64, 132]. This type of behavior is thought to be generic for nonlinear dispersive PDEs and this leads to the *(Soliton) Resolution Conjecture*, which (vaguely formulated) states that any global solution of a nonlinear dispersive PDE will eventually decompose at large time as a combination of non-scattering structures (*e.g.* a sum of solitary waves) and a radiative term.

Until recently, such conjecture had only been established for some integrable models, *e.g.* the Korteweg-de Vries equation. The breakthrough approach introduced by Duyckaerts, Kenig and Merle allowed to prove this conjecture for some non-integrable equations such as the energy-critical wave equation [63] or the equivariant wave maps to the sphere [52]. It remains an open problem for most of the classical nonlinear dispersive equations.

The Soliton Resolution Conjecture motivates the study of multi-soliton solutions for nonlinear dispersive PDE, *i.e.* solutions which behave at large time as a sum of solitons. Indeed, investigating the existence and properties of solutions of dispersive equations made of a combination of non-scattering structures is a first step toward a proof of a Decomposition Conjecture, and multi-solitons are one of the simplest examples of a combination of non-scattering structures.

Several methods are available to obtain multi-solitons. They have been first constructed for NLS in the one dimensional cubic focusing case by Zakharov and Shabat [161] using the inverse scattering transform method (IST). The IST is a powerful tool to study nonlinear dispersive equations and to exhibit non-trivial nonlinear dynamics for these equations. However, the IST application is restricted to equations which are completely integrable, like for example the Korteweg-de Vries equation and the cubic nonlinear Schrödinger equation in dimension 1. Moreover, integrability probably does not hold for (3.1.1).

Another method to construct multi-soliton solutions of non-integrable equations was introduced by Merle [119] for the L^2 critical NLS, and then extended to the L^2 subcritical gKdV (by Martel [115]), to the L^2 subcritical NLS (by Martel and Merle [116]). This method uses tools usually called *energy techniques*, in the sense that it relies on the use of the second variation of the energy as a Lyapunov functional to control the difference of a solution u with the soliton sum R. It was later fine-tuned to allow the treatment of L^2 supercritical equations [55] and of profiles made with excited states [53]. One can also cite the works on the non-linear Klein-Gordon equation by Côte and Muñoz [56] and Côte and Martel [54], and on the stability of multi-solitons for generalized Korteweg-de Vries equations and L^2-subcritical nonlinear Schrödinger equations from Martel, Merle and Tsai [117, 118].

In this article, we show that this method for the construction of multi-solitons can be extended to a focusing logarithmic nonlinearity, revealing the existence of *multi-Gaussons* for (3.1.1). We will denote by $\mathcal{F}$ the Fourier Transform so that

$$\mathcal{F}(H^1)(\mathbb{R}^d) = \left\{ v \in L^2(\mathbb{R}^d), \| |x| \, v \|_{L^2} < \infty \right\}$$

is a Hilbert space with its usual scalar product.

Theorem 3.1.11 (Existence of multi-Gaussons)**.** *Consider $\lambda > 0$, $N \in \mathbb{N}^*$, $d \in \mathbb{N}^*$ and take $(v_k)_{1\leq k\leq N}$ and $(x_k)_{1\leq k\leq N}$ two families in $\mathbb{R}^d$, $(\omega_k)_{1\leq k\leq N}$ and $(\theta_k)_{1\leq k\leq N}$ two families of real numbers. Define*

$$v_* := \min_{j\neq k} |v_j - v_k|, \qquad G_k := G^d_{\omega_k, x_k, v_k, \theta_k}.$$

If $v_* > 0$, then there exist a unique solution $u \in \mathcal{C}_b(\mathbb{R}, W(\mathbb{R}^d)) \cap L^\infty_{\text{loc}}(\mathbb{R}, \mathcal{F}(H^1)(\mathbb{R}^d))$ to (3.1.1) *and $T \in \mathbb{R}$ such that $\forall t \geq 0$,*

$$\left\| u(T+t) - \sum_{k=1}^N G_k(T+t) \right\|_{H^1\cap\mathcal{F}(H^1)} \leq e^{-\frac{\lambda (v_* t)^2}{4}}. \tag{3.1.7}$$

In particular, there exists $C > 0$ (depending on λ, v_* and T) such that

$$\left\| u(t) - \sum_{k=1}^N G_k(t) \right\|_{H^1\cap\mathcal{F}(H^1)} \leq C\, e^{-\frac{\lambda (v_* t)^2}{8}}, \qquad \forall t \geq 0.$$

Several features are new for this case and should be pointed out.

First, for NLS with a nonlinearity of the form $g(|u|^2)\,u$, the nonlinearity should usually satisfy a flatness property at 0 for such a result (for example in [53]: $g(0) = 0$ and $\lim_{s\to 0} s\, g'(s) = 0$). Here, the nonlinearity in (3.1.1) is not flat at all at 0: even more, it is not even defined at 0 since we gave here $g = -\lambda \ln$, and

$$g(s) \underset{s\to 0^+}{\longrightarrow} +\infty, \qquad s\, g'(s) = -\lambda \quad \forall s > 0.$$

The convergence rate of the solution to the sum of solitons is also very interesting. The convergence rate for NLS with polynomial-like nonlinearity is exponential, whereas it is "Gaussian-like" here, which is much faster. Moreover, it does not depend on the frequencies of the solitons anymore (even though T does), unlike in [53].

Such features may be surprising at first sight. However, they can be explained by the decay at infinity of the Gaussons. Indeed, in the same way as for the convergence rate, the decay of the solitons at infinity is usually exponential, with a rate depending on its frequency, whereas the solitons for (3.1.1) are the Gaussons, in particular Gaussian functions, whose decay at infinity is much faster and independent of their frequencies (up to a multiplicative constant). Moreover, the nonlinearity is still smooth enough: it is smooth far from the vacuum and, near the vacuum, $u \ln |u|^2$ remains almost lipschitz.

Remark also that the convergence is in $H^1(\mathbb{R}^d) \cap \mathcal{F}(H^1(\mathbb{R}^d))$. Hence the same convergence rate holds in the energy space $W(\mathbb{R}^d)$ since

$$H^1(\mathbb{R}^d) \cap \mathcal{F}(H^1(\mathbb{R}^d)) \subset W(\mathbb{R}^d).$$

Indeed, for all $\varepsilon > 0$, there holds $\left| |f|^2 \ln|f|^2 \right| \lesssim |f|^{2-\varepsilon} + |f|^{2+\varepsilon}$, which gives the inclusion for ε small enough (depending on the dimension d). We refer to [33, 67] for some more details on the computation. We can compare this convergence to the case of a subcritical nonlinearity, where the convergence is also proved in the energy space for such an equation, *i.e.* "only" H^1.

Remark 3.1.12. Theorem 3.1.11 does not say that, as soon as we fixed all the parameters, the multi-Gausson is unique. However, there is a unique multi-Gausson which satisfies the convergence rate property (3.1.7) (for those parameters). Indeed, any other multi-Gausson v would satisfy for all $t \geq T_1$

$$\left\| v(t) - \sum_{k=1}^N G_k(t) \right\|_{L^2} \geq C_1\, e^{-2\lambda t}.$$

for some constant $C_1 > 0$ and some time T_1 (see Lemma 3.6.1). Therefore, it is a *rigidity* property.

Existence of multi-breathers and multi-gaussians. The Gaussons are not the only non-scattering structures that we are aware of for this equation: we have excited states, but we also have breathers and more generally gaussian solutions. Thus, the (Soliton) Resolution Conjecture also motivates the study of multi-gaussians. However, those breathers and gaussian solutions are *not* bound state for the energy E, and then the same energy techniques cannot be applied. Nevertheless, the method used to find the L^2 estimate for the multi-Gaussons for (3.1.1) does not involve the energy: such a method can be tuned in order to fit with these multi-gaussians.

Theorem 3.1.13 (Existence of multi-gaussians)**.** *Consider $\lambda > 0$, $N \in \mathbb{N}^*$, $d \in \mathbb{N}^*$ and take $(v_k)_{1\leq k\leq N}$ and $(x_k)_{1\leq k\leq N}$ two families in $\mathbb{R}^d$, $(\omega_k)_{1\leq k\leq N}$ and $(\theta_k)_{1\leq k\leq N}$ two families of real numbers, and $(A_k^{\text{in}})_{1\leq k\leq N}$ a sequence of complex matrices in $S_d(\mathbb{C})^{\text{Re}+}$. Define $A_k(t)$ the solution to* (3.1.4) *with initial data A_k^{in} and*

$$v_* := \min_{j\neq k} |v_j - v_k|, \qquad B_k := B^{A_k^{\text{in}}}_{\omega_k, x_k, v_k, \theta_k}, \qquad \sigma_- := \frac{1}{2} \inf_{t,k} \sigma(\operatorname{Re} A_k(t)) > 0.$$

If $v_* > 0$, then there exist a unique solution $u \in \mathcal{C}_b(\mathbb{R}, W(\mathbb{R}^d))$ to (3.1.1) *and $T \in \mathbb{R}$ such that $\forall t \geq 0$,*

$$\left\| u(T+t) - \sum_{k=1}^{N} B_k(T+t) \right\|_{L^2} \leq e^{-\frac{\sigma_-(v_* t)^2}{4}}. \tag{3.1.8}$$

In particular, there exists $C > 0$ (depending on λ, v_* and T) such that

$$\left\| u(t) - \sum_{k=1}^{N} B_k(t) \right\|_{L^2} \leq C\, e^{-\frac{\sigma_-(v_* t)^2}{8}}, \qquad \forall t \geq 0.$$

Remark 3.1.14. Remark that the convergence is only in L^2 norm here, unlike the previous theorem for multi-Gaussons where the convergence is in $H^1 \cap \mathcal{F}(H^1)$. However, we do believe that a convergence in $H^1 \cap \mathcal{F}(H^1)$ should hold, but the energy techniques for such a proof do not hold, as already pointed out.

Remark 3.1.15. Again, the "uniqueness" of the multi-gaussians is subjected to the convergence rate property (3.1.8), but any other solution v would satisfy for all $t \geq T_1$

$$\left\| v(t) - \sum_{k=1}^{N} B_k(t) \right\|_{L^2} \geq C_1 e^{-2\lambda t}.$$

for some constant $C_1 > 0$ and some time T_1 in the same way (see Lemma 3.6.1).

3.1.4. Scheme of the proof and outline

Our strategy for the proofs of Theorems 3.1.11 and 3.1.13 is inspired from the works [116, 119, 55, 53]: we take a sequence of time $T_n \to +\infty$ and a set of final data $u_n(T_n) = B(T_n)$ where $B := \sum_{k=1}^{N} B_k$ (in the case of multi-Gaussons, $A_k^{\text{in}} = 2\lambda I_d$ like already pointed out in Remark 3.1.9). Our goal is to prove that the solutions u_n to (3.1.1) (which approximate a multi-soliton) enjoy uniform $H^1(\mathbb{R}^d)$ and $\mathcal{F}(H^1(\mathbb{R}^d))$ (in the case of multi-Gaussons) or only $L^2(\mathbb{R}^d)$ (in the more general case) decay estimates on $[T, T_n]$ for some T independent of n. Then, a compactness in L^2 is proved which shows that (u_n) (up to a subsequence) converges to a multi-soliton solution to (3.1.1).

For multi-Gaussons, like in [116, 55, 53], the uniform backward $H^1(\mathbb{R}^d)$-estimates rely on slow variation of localized conservation laws as well as an $H^1(\mathbb{R}^d)$ coercivity of the action around G up to an $L^2(\mathbb{R}^d)$-norm and is proved through a bootstrap property. The main new difficulty for (3.1.1) compared to a subcritical polynomial-like nonlinearity is that the energy E on which the action is constructed is not of class $\mathcal{C}^2$ because of the potential energy. However, a weaker Taylor expansion holds, and thus the coercivity (in $H^1(\mathbb{R}^d)$ up to an $L^2(\mathbb{R}^d)$ norm) can still be proved.

Another new feature of the proof is that the uniform $L^2(\mathbb{R}^d)$-estimates can be found thanks to a more direct computation from [69], inspired from [40, 33], performed in Section 3.2 (both for the multi-gaussian and the multi-Gausson cases). Thus, for the Gaussons case, the bootstrap property is needed only for the homogeneous $\dot{H}^1(\mathbb{R}^d)$-norm. After proving the uniform H^1-estimates property in Section 3.3, a similar improved computation as that for the uniform $L^2(\mathbb{R}^d)$-estimate is performed for the uniform $\mathcal{F}(H^1(\mathbb{R}^d))$ estimates in Section 3.4, which also gives compactness in $L^2(\mathbb{R}^d)$.

As for the multi-gaussian case, the compactness property is proved in Section 3.5 thanks to a cut-off argument, similar to that in [116, 55, 53]. Eventually, the rigidity property for both cases is proved in Section 3.6 thanks to a consequence of the L^2 energy estimate (Lemma 3.1.3).

Notation

From now on, C will denote a positive constant which does not depend on anything and C_0 a positive constant which is independent of the time and of n (but may depend on other parameters). They also may change from line to line. Moreover, all the functional spaces in space are in $\mathbb{R}^d$, which will be implicit. For instance, we will denote by L^2, H^1, $\mathcal{F}(H^1)$ and W instead of $L^2(\mathbb{R}^d)$, $H^1(\mathbb{R}^d)$, $\mathcal{F}(H^1(\mathbb{R}^d))$ and $W(\mathbb{R}^d)$. Furthermore, all the integrals will be in $\mathbb{R}^d$ and all the sums will be from 1 to N, except if indicated.

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3.2. Uniform L^2-estimates

3.2.1. Approximate solutions and convergence toward a multi-soliton

As already said, the proof follows the general scheme laid down by Martel, Merle and Tsai [117] for the Korteweg-de Vries equation and adapted by Martel and Merle [116] in the case of nonlinear Schrödinger equation (we can also cite for instance the works [119, 55, 53]). We choose an increasing sequence of times $(T_n)_{n\in\mathbb{N}}$ with $T_n \to \infty$ as $n \to \infty$ and we define solutions u_n to (3.1.1) with *final data* $u_n(T_n) = B(T_n)$ where $B := \sum B_k$ with $B_k := B^{A_k^{\text{in}}}_{\omega_k,x_k,v_k,\theta_k}$. In the Gaussons case, we take $A_k^{\text{in}} = 2\lambda I_d$ so that $B^{A_k^{\text{in}}}_{\omega_k,x_k,v_k,\theta_k} = G^d_{\omega_k,x_k,v_k,\theta_k}$ for all k (see Remark 3.1.9), and we may also say G_k instead of B_k and G instead of B. Thanks to Theorem 3.1.2, we know that all u_n are well-defined and global since $B(T_n)$ is obviously in W, so $u_n \in \mathcal{C}_b(\mathbb{R}, W)$.

Our goal is to prove that u_n (called *approximate multi-gaussian*, resp. *approximate multi-soliton* in the Gausson case) converges (up to a subsequence) to u, a multi-gaussian (resp. multi-soliton) solution to (3.1.1) which satisfies the estimates of Theorem 3.1.13 (resp. 3.1.11). For this, we prove that the u_ns satisfy the same kind of decay estimate as in (3.1.8) (resp. (3.1.7)) but only up to T_n.

Multi-gaussian case. As in [116, 53], Theorem 3.1.13 relies on two important propositions:

Proposition 3.2.1 (Uniform Estimates)**.** *There exists $T \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ such that $T_n > T$ and for any $t \in [0, T_n - T]$,*

$$\|u_n(T+t) - B(T+t)\|_{L^2} \leq e^{-\frac{\sigma_-(v_* t)^2}{4}}. \tag{3.2.1}$$

Proposition 3.2.2 (Compactness)**.** *There exists $u_{\text{in}} \in W$ such that (up to a subsequence)*

$$\lim_{n\to\infty} \|u_n(T) - u_{\text{in}}\|_{L^2} = 0.$$

Theorem 3.1.13 can then be easily proved thanks to Propositions 3.2.1 and 3.2.2.

Proof of Theorem 3.1.13. Let u_{in} given by Proposition 3.2.2. Take the subsequence of (u_n) (still denoted (u_n)) such that $(u_n(T))$ converges to u_{in} in L^2 as $n \to \infty$ and let $u \in \mathcal{C}_b(\mathbb{R}, W)$ be the solution to (3.1.1) with initial data $u(T) = u_{\text{in}} \in W$ given by Theorem 3.1.2. Then, thanks to Lemma 3.1.3, $\mu_n := u - u_n$ satisfies for all $t \geq 0$ and $n \in \mathbb{N}$,

$$\|\mu_n(T+t)\|_{L^2} \leq \|\mu_n(T)\|_{L^2}\, e^{2\lambda t}.$$

By definition of μ_n and u, $\mu_n(T) = u_n(T) - u_{\text{in}} \to 0$ in L^2 as $n \to \infty$. Therefore, for all $t \geq 0$, the previous inequality shows that $u_n(T+t) \to u(T+t)$ in L^2 as $n \to \infty$. Then, using Proposition 3.2.1 and taking the limit $n \to \infty$ for any $t \geq 0$, we get:

$$\|u(T+t) - B(T+t)\|_{L^2} \leq \lim_{n\to\infty} \|u_n(T+t) - B(T+t)\|_{L^2} \leq e^{-\frac{\sigma_-(v_* t)^2}{4}}. \qquad \square$$

The Uniform Estimates property 3.2.1 will be proved in Subsection 3.2.2, while Section 3.5 will be devoted to the proof of the Compactness property 3.2.2. However, before that, we also look at the multi-soliton case.

Multi-soliton case. For the multi-soliton case, the same kind of properties will be proved, but the uniform estimates are in $H^1 \cap \mathcal{F}(H^1)$ norm, not only L^2.

Proposition 3.2.3 (Uniform Estimates)**.** *There exists $T \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ such that $T_n > T$ and for any $t \in [0, T_n - T]$,*

$$\|u_n(T+t) - G(T+t)\|_{H^1\cap\mathcal{F}(H^1)} \leq e^{-\frac{\lambda(v_* t)^2}{4}}. \tag{3.2.2}$$

Like for the multi-gaussian case, we also need a compactness property. However, it obviously results from the uniform estimates:

Corollary 3.2.4 (Compactness)**.** *There exists $u_{\text{in}} \in H^1 \cap \mathcal{F}(H^1)$ such that (up to a subsequence)*

$$\lim_{n\to\infty} \|u_n(T) - u_{\text{in}}\|_{L^2} = 0.$$

Proof of Corollary 3.2.4. $u_n(T)$ is uniformly bounded in $H^1 \cap \mathcal{F}(H^1)$ by taking $t = 0$ in (3.2.2), which yields the conclusion since the embedding $H^1 \cap \mathcal{F}(H^1) \subset L^2$ is compact (see for example [128, Theorem XIII.67]). $\square$

Remark 3.2.5. Unlike this case, the compactness property is not as obvious in [116, 53] (or for the multi-gaussian case above). Indeed, the uniform estimates are only in H^1 there (or even in L^2 for the multi-gaussian), whereas we also have $\mathcal{F}(H^1)$ here. Thus, the authors had to prove in there a uniform equicontinuity of the sequence (u_n) by using a cut-off argument and the uniform estimates in L^2. The same kind of proof will be used for Proposition 3.2.2.

Theorem 3.1.11 is then a corollary of Proposition 3.2.3 and Corollary 3.2.4. Its proof is totally similar to the proof of Theorem 3.1.13, using the weakly lower semi-continuity of the $H^1 \cap \mathcal{F}(H^1)$-norm. For this case, the uniform estimates in L^2 will also be proved in Subsection 3.2.2, in the same time as for the multi-gaussian case. As for the H^1 and the $\mathcal{F}(H^1)$ uniform estimates, they will be proved in Sections 3.3 and 3.4 respectively.

3.2.2. Uniform L^2-estimates

In [116, 55, 53], the uniform estimates in H^1 for multi-solitons are found thanks to a bootstrap argument. Here, the uniform estimates in L^2 can be found without it, directly with a rough stability estimate thanks to the strong decay of our solitons at infinity. This method can also be extended to the gaussian case, and we prove both cases for Proposition 3.2.1.

Remark 3.2.6. We recall that the multi-soliton case is a particular case of multi-gaussian where we have taken $A_k^{\text{in}} = 2\lambda I_d$, so that $A_k(t) = 2\lambda I_d$. Thus, by definition of σ_-, we get $\sigma_- = \lambda$, which gives exactly the expected convergence rate for multi-Gausson, at least for the L^2 norm.

The computation used for this estimate is almost the same computation as that in [69]. Directly inspired from the computation for the energy estimate in L^2 found in [40], it is the consequence of the following lemma:

Lemma 3.2.7 ([40, Lemma 1.1.1])**.** *There holds*

$$\left|\operatorname{Im}\left((z_2 \ln|z_2|^2 - z_1 \ln|z_1|^2)(\overline{z_2} - \overline{z_1})\right)\right| \le 2|z_2 - z_1|^2, \qquad \forall z_1, z_2 \in \mathbb{C}.$$

Indeed, for any integer $N \ge 1$ and any solutions $v, v_1, \dots, v_N$ to (3.1.1), the function $w := v - V$ with $V := \sum v_j$ satisfies:

$$i\,\partial_t w + \frac{1}{2}\Delta w = -\lambda\Big(v \ln|v|^2 - \sum v_j \ln|v_j|^2\Big).$$

Therefore, there holds

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{L^2}^2 &= -\lambda \operatorname{Im}\int \Big(v\ln|v|^2 - \sum v_j \ln|v_j|^2\Big)(\overline{v} - \overline{V})\,\mathrm{d}x \\
&= -\lambda \operatorname{Im}\int \Big(v\ln|v|^2 - V\ln|V|^2\Big)(\overline{v} - \overline{V})\,\mathrm{d}x - \lambda \operatorname{Im}\int \Big(V\ln|V|^2 - \sum v_j \ln|v_j|^2\Big)(\overline{v} - \overline{V})\,\mathrm{d}x \\
\frac{1}{2}\left|\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{L^2}^2\right| &\le 2|\lambda|\,\|w\|_{L^2}^2 + |\lambda|\int \left|V\ln|V|^2 - \sum v_j \ln|v_j|^2\right||w|\,\mathrm{d}x \\
&\le 2|\lambda|\,\|w\|_{L^2}^2 + |\lambda|\left\|V\ln|V|^2 - \sum v_j \ln|v_j|^2\right\|_{L^2}\|w\|_{L^2}.
\end{aligned}$$

Dividing by $\|w\|_{L^2}$, we obtain the inequality:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{L^2}\right| \le 2|\lambda|\,\|w\|_{L^2} + |\lambda|\left\|V\ln|V|^2 - \sum_{j=1}^{N} v_j \ln|v_j|^2\right\|_{L^2}. \tag{3.2.3}$$

The core of the problem is to estimate the last term in order to be able to perform a backward Gronwall lemma. In the case where $v_j = B_j$, this is an explicit term which can be estimated thanks to [69, Lemma 3.2] that we recall here.

Lemma 3.2.8. *For any $d \in \mathbb{N}^*$, there exists $C_d > 0$ such that the following holds. Let $N \in \mathbb{N}^*$ and take $x_k \in \mathbb{R}^d$, $\omega_k \in \mathbb{R}$, $\Lambda_k \in S_d(\mathbb{C})^{\mathrm{Re}+}$ and $\theta_k : \mathbb{R}^d \to \mathbb{R}$ a real measurable function for $k = 1, \dots, N$, and define for all $x \in \mathbb{R}^d$*

$$g_k(x) = \exp\left[i\theta_k(x) + \omega_k - (x - x_k)^\top \Lambda_k (x - x_k)\right],$$

as well as

$$g(x) = \sum_{k=1,\dots,N} g_k(x).$$

If

$$\varepsilon := \left(\min_{k\neq j}|x_j - x_k|\right)^{-1} < \varepsilon_0 := \min\left(\frac{\sqrt{\lambda_+}}{\max(\sqrt{\delta\omega + 1}, \sqrt{\ln N})}, \sqrt{\frac{\lambda_-}{d+2}}\right)$$

where $\delta\omega := \max_{j,k}|\omega_k - \omega_j|$, $\lambda_+ = \max_k \operatorname{Re}\sigma(\Lambda_k)$ and $\lambda_- = \min_k \operatorname{Re}\sigma(\Lambda_k) > 0$, then

$$\left\|g\ln|g| - \sum_{k=1}^{N} g_k \ln|g_k|\right\|_{L^2} \le C_d N^{\frac{3}{2}}\,\frac{\lambda_+}{\varepsilon^{\frac{d}{2}+1}\sqrt{\lambda_-}}\,\exp\left[-\frac{\lambda_-}{4\varepsilon^2} + \max_j \omega_j\right]. \tag{3.2.4}$$

To be able to apply Lemma 3.2.8, we need the gaussians to be "well separated" (which is given by the condition $\varepsilon < \varepsilon_0$). But this happens, eventually after some time

$$T_{\text{sep}} := \max_{k,j}\frac{\varepsilon_0^{-1} + |x_k - x_j|}{v_*},$$

where ε_0 is defined in Lemma 3.2.8. Indeed, for all times $t \geq 0$, there holds

$$|x_j + v_j(T_{\text{sep}} + t) - (x_k + v_k(T_{\text{sep}} + t))| \geq \varepsilon_0^{-1} + v_* t. \tag{3.2.5}$$

Then, we will also need to estimate the Gauss error function, which can be done in the usual way.

Lemma 3.2.9. *For all $n \in \mathbb{N}$ and $d \in \mathbb{N}^*$, there exists $C_{d,n} > 0$ such that for all $\gamma > 0$ and $R \geq \gamma^{-\frac{1}{2}}$, there holds*

$$M_n := \int_{\mathcal{B}_R^\complement} |x|^n e^{-\gamma|x|^2} \, \mathrm{d}x \leq C_{d,n} \frac{R^{d+n-2}}{\gamma} e^{-\gamma R^2},$$

where $\mathcal{B}_R = \mathcal{B}_{\mathbb{R}^d}(0, R)$.

We postpone the proof of this lemma to 3.A and we now prove Proposition 3.2.1.

Proof of Proposition 3.2.1. Thanks to (3.2.5) and the fact that $0 < \sigma_- = \inf_{t,k} \sigma(\operatorname{Re} A_k(t)) \leq \sup_{t,k} \sigma(\operatorname{Re} A_k(t)) < +\infty$ with Proposition 3.1.4, we can apply Lemma 3.2.8 : for all $t > 0$,

$$\begin{aligned} &\left\| B(T_{\text{sep}} + t) \ln|B(T_{\text{sep}} + t)| - \sum_{k=1}^{N} B_k(T_{\text{sep}} + t) \ln|B_k(T_{\text{sep}} + t)| \right\|_{L^2(\mathbb{R})} \\ &\qquad \leq C_0 (\varepsilon_0^{-1} + v_* t)^{\frac{d+2}{2}} \exp\left[-\frac{\sigma_-(\varepsilon_0^{-1} + v_* t)^2}{4} \right] \leq C_0 \exp\left[-\frac{\sigma_-(v_* t)^2}{4} \right]. \end{aligned} \tag{3.2.6}$$

Take n large enough so that $T_n > T_{\text{sep}}$ and set $w_n := u_n - B$. (3.2.3) gives that for all $t > 0$,

$$\left| \partial_t \|w_n(T_{\text{sep}} + t)\|_{L^2} \right| \leq 2\lambda \|w_n(T_{\text{sep}} + t)\|_{L^2} + C_0 \exp\left[-\frac{\sigma_-(v_* t)^2}{4} \right].$$

The Gronwall lemma (backward in time) between t and $T_n - T_{\text{sep}}$ for $0 \leq t \leq T_n - T_{\text{sep}}$ and the fact that $w_n(T_n) = 0$ yields

$$\begin{aligned} \|w_n(T_{\text{sep}} + t)\|_{L^2} &\leq C_0 \int_t^{T_n} \exp\left[-\frac{\sigma_-(v_* s)^2}{4} + 2\lambda(s - t) \right] \mathrm{d}s \\ &\leq C_0 \, e^{-2\lambda t} \int_t^{\infty} \exp\left[-\frac{\sigma_-(v_* s)^2}{4} + 2\lambda s \right] \mathrm{d}s. \end{aligned}$$

Then, we estimate the integral.

$$\begin{aligned} \int_t^{\infty} \exp\left[-\frac{\sigma_-(v_* s)^2}{4} + 2\lambda s \right] \mathrm{d}s &= \int_t^{\infty} \exp\left[-\sigma_- \left(\frac{v_* s}{2} - \frac{2\lambda}{\sigma_- v_*} \right)^2 + \frac{4\lambda^2}{\sigma_- v_*^2} \right] \mathrm{d}s \\ &= \frac{2}{v_*} \int_{\tilde{t}}^{\infty} \exp\left[-\sigma_- r^2 + \frac{4\lambda^2}{\sigma_- v_*^2} \right] \mathrm{d}r, \end{aligned}$$

where $\tilde{t} := \frac{v_* t}{2} - \frac{2\lambda}{\sigma_- v_*}$. Using Lemma 3.2.9 with $n = 0$ and $d = 1$, we get for all t such that $\tilde{t} = \frac{v_* t}{2} - \frac{2\lambda}{\sigma_- v_*} > 0$

$$\int_t^{\infty} \exp\left[-\frac{\sigma_-(v_* s)^2}{4} + 2\lambda s \right] \mathrm{d}s \leq \frac{C_0}{\tilde{t}} \exp\left[-\sigma_- \tilde{t}^2 + \frac{4\lambda^2}{\sigma_- v_*^2} \right] = \frac{C_0}{\tilde{t}} \exp\left[-\frac{\sigma_-(v_* t)^2}{4} + 2\lambda t \right]$$

Hence, setting $t_1 := \frac{4}{v_*^2}$, we have for all n large enough and for all $t \in [t_1, T_n - T_{\text{sep}}]$

$$\|w_n(T_{\text{sep}} + t)\|_{L^2} \leq \frac{C_0}{t - t_1} \exp\left[-\frac{\sigma_-(v_* t)^2}{4} \right],$$

which leads to the result by taking $T' > T_{\text{sep}} + t_1$ large enough. ∎

3.3. UNIFORM H^1-ESTIMATES

This section (and so is Section 3.4) is completely devoted to the multi-soliton case, *i.e.* Proposition 3.2.3, so that here $B_j = G_j$ are all Gaussons. However, since the Gausson is a particular case of gaussian solutions, the previous section holds. Moreover, for Gaussons, we have $A_j(t) = 2\lambda I_d$, thus $\sigma_- = \lambda$ here. Therefore, Proposition 3.2.1 gives some $T' \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ such that $T_n > T'$ and for any $t \in [0, T_n - T']$,

$$\|u_n(T'+t) - G(T'+t)\|_{L^2} \leq e^{-\frac{\lambda(v_* t)^2}{4}}. \tag{3.3.1}$$

The second step in order to prove Proposition 3.2.3 is to get the uniform estimates in H^1. Since we already have it for L^2, we need to prove it only for $\dot{H}^1$.

Proposition 3.3.1. *There exists $T > T'$ such that for all $n \in \mathbb{N}$ such that $T_n > T$ and for any $t \in [0, T_n - T]$,*

$$\|u_n(T+t,.) - G(T+t,.)\|_{\dot{H}^1} \leq e^{-\frac{\lambda(v_* t)^2}{4}}. \tag{3.3.2}$$

The proof relies on a bootstrap argument. Indeed, from the definition of the final data $u_n(T_n)$ and continuity of u_n in time with values in H^1, it follows that (3.3.2) holds on an interval $[t^\dagger, T_n]$ for $t^\dagger$ close enough to T_n. Then the following Proposition 3.3.2 shows that we can actually improve it to a better estimate, hence leaving enough room to extend the interval on which the original estimate holds.

Proposition 3.3.2 (Bootstrap Property)**.** *There exists $T'' > T'$ and $t_* > 0$ such that for all $n \in \mathbb{R}$ such that $T_n > T'' + t_*$ and for all $t^\dagger \in [t_*, T_n - T'']$, the following holds. If for all $t \in [t^\dagger, T_n - T'']$ we have*

$$\|u_n(T''+t) - G(T''+t)\|_{\dot{H}^1} \leq e^{-\frac{\lambda(v_* t)^2}{4}}, \tag{3.3.3}$$

then for all $t \in [t^\dagger, T_n - T'']$ there holds

$$\|u_n(T''+t) - G(T''+t)\|_{\dot{H}^1} \leq \frac{1}{2} e^{-\frac{\lambda(v_* t)^2}{4}}.$$

Remark 3.3.3. To be more precise, what we prove is the following: for all $t^\dagger \in [0, T_n - T'']$, if for all $t \in [t^\dagger, T_n - T'']$ we have (3.3.3), then for all $t \in [t^\dagger, T_n - T'']$ there holds

$$\|u_n(T''+t) - G(T''+t)\|_{\dot{H}^1}^2 \leq C_0\, t^{-1} e^{-\frac{\lambda(v_* t)^2}{2}}.$$

Therefore, we need some $t_* > 0$ so that $\frac{C_0}{t} \leq \frac{1}{4}$ for all $t \geq t_*$ in order to have a true bootstrap argument.

In [116, 55, 53], the bootstrap argument to get the uniform estimates in H^1 uses a modified action defined with localized quantities (localization of the conserved quantities around each member/soliton). It is known that each soliton gives only at most few negative L^2 "bad directions" in the Hessian of the action and has an exponential decay at infinity. Therefore the Hessian of this modified action is "almost coercive", up to an exponentially decreasing function of time, and to some bad directions in L^2 which are controlled either by modulation ([116] for instance) or by controlling the L^2-growth without the help of the Hessian ([53] for instance). Therefore one gets a slightly better estimate in H^1 than provided by the assumption.

Here, such an argument cannot be used directly: the energy is not $\mathcal{C}^2$ because of the appearance of a $\ln|v|$ term from the potential energy, which is ill-posed in $\mathcal{L}(L^2)$, and therefore we cannot have a Taylor expansion relation between the modified action and its linearized. However, a weaker expansion of the potential energy (coming from Lemma 3.3.18) can still be used, giving enough room to get an H^1-coercivity up to an (almost) L^2 norm, as shown in Lemma 3.3.8.

From this proposition, the proof of Proposition 3.3.1 is then completely similar to those in [116, 53], and we refer to them for more precision.

3.3.1. The Bootstrap Property

The idea of the proof of the Bootstrap Property 3.3.2 is similar to that of [53], which is reminiscent of the technique used to prove stability for a single soliton in the subcritical case (see *e.g.* [151, 153, 84, 85, 106]). Indeed, it is known that the solitons G_j are critical points and even ground states respectively for the action functionals

$$S_j(v) := E(v) + \Big(2\lambda\omega_j + \frac{|v_j|^2}{2}\Big) M(v) - v_j \cdot \mathcal{J}(v).$$

In the previous articles, the Hessian of these functionals is coercive on a subspace of H^1 of finite co-dimension in L^2. At large time, the components of the multi-soliton are well-separated and thus the analysis can be localized around each soliton to gain an

H^1-local control, up to a space of finite dimension in L^2, for the linearized. Thus the sum of the localized functionals is (locally around G) H^1-coercive up to some negligible terms and few directions in L^2, which can be controlled either by modulation of the invariants (for ground states) or by a better control of the L^2-norm.

However, the fact of having few bad L^2 directions is not necessary, in particular since we already have an L^2-estimate thanks to (3.3.1): the constructed functional S^{loc}, which is time-dependent and "slowly varies" for all u_n (in a sense explained later), only need to satisfy the fact that there exists some $K_1 > 0$ and $K_2 \in \mathbb{R}$ such that for all t large enough and for all $v \in W$ (possibly near $G(t)$)

$$S(t,v) - \sum_j S_j(G_j) \geq K_1 \|w\|_{H^1}^2 - K_2 \|w\|_{L^2}^2,$$

with $w := v - G(t)$.

This functional S^{loc} is constructed in a similar way as in the above articles. It requires the use of an action-like functional, defined with quantities localized around each Gausson. For this, we need a localization procedure. However, unlike in [116, 53] for example, we will not localize only in one dimension by taking a particular direction in which all the speed components are well ordered. Indeed, such a tactics would lead to a slower convergence rate, since the minimal relative speed in this direction would be smaller.

Take some $T'' > T'$ and $t_* > 0$ to be fixed later and take any $n \in \mathbb{N}$ such that $T_n > T'' + t_*$. Set again $w_n := u_n - G$. Let $t^\dagger \in [t_*, T_n - T'']$ and assume that for all $t \in [t^\dagger, T_n - T'']$ there holds

$$\|\nabla w_n(t')\|_{L^2} \leq e^{-\frac{\lambda (v_* t)^2}{4}}, \tag{3.3.4}$$

with the notation $t' := T'' + t$. By (3.3.1), we know that there also holds

$$\|w_n(t')\|_{L^2} \leq e^{-\frac{\lambda (v_* (t+\tau))^2}{4}}, \tag{3.3.5}$$

where $\tau := T'' - T'$.

Take $\phi : \mathbb{R} \to \mathbb{R}$ a $\mathcal{C}^\infty$ function such that $\phi(s) = 1$ for all $s \leq -1$, $\phi(s) = 0$ for all $s \geq 1$ and $\phi(s) \in [0,1]$ and $-1 \leq \phi'(s) \leq 0$ for all $s \in \mathbb{R}$. Then we define for all $t \geq 0$ (still with the notation $t' = T'' + t$ for the delayed time):

— the center of each Gausson

$$x_j^*(t') := x_j + t'\, v_j,$$

— functions ψ_j ($1 \leq j \leq N$) with the j-th member weighted around the j-th Gausson:

$$\psi_j(t', x) := \phi\Big(\big|x - x_j^*(t')\big| - \frac{v_* t}{2} - 2\Big) \qquad \forall x \in \mathbb{R}^d, \text{ for } j = 1, \dots, N.$$

— a last function ψ_0 completing the previous family so that $(\psi_j)_{0 \leq j \leq N}$ is a partition of unity:

$$\psi_0 := 1 - \sum_{j=1}^N \psi_j.$$

First, we need to prove that (ψ_j) is really a smooth partition of unity. The only two things which still must be proved are the facts that $\psi_0 \geq 0$ and that all the ψ_js are smooth.

Lemma 3.3.4. *If T'' is large enough, then for all $1 \leq j < k \leq N$ and all $t \geq 0$,* $\operatorname{supp} \psi_j(t') \cap \operatorname{supp} \psi_k(t') = \emptyset$.

Proof. By definition of ϕ and ψ_j (for $j \geq 1$), we know that for all $t \geq 0$,

$$\operatorname{supp} \psi_j(t') \subset \mathcal{B}\Big(x_j^*(t'), \frac{v_* t}{2} + 3\Big)$$

Thus, there holds for all $1 \leq j < k \leq N$ and all $t \geq 0$

$$\operatorname{supp} \psi_j(t') \cap \operatorname{supp} \psi_k(t') \subset \mathcal{B}\Big(x_j^*(t'), \frac{v_* t}{2} + 3\Big) \cap \mathcal{B}\Big(x_k^*(t'), \frac{v_* t}{2} + 3\Big) = \emptyset,$$

since (3.2.5) and $T' > T_{\text{sep}}$ lead to

$$\big|x_j^*(t') - x_k^*(t')\big| \geq \varepsilon_0^{-1} + v_*(t+\tau) > v_* t + 6 \tag{3.3.6}$$

as soon as τ (*i.e.* T'') is large enough. ☐

Thus, we now assume that T'' is large enough so that the previous result holds. Then:

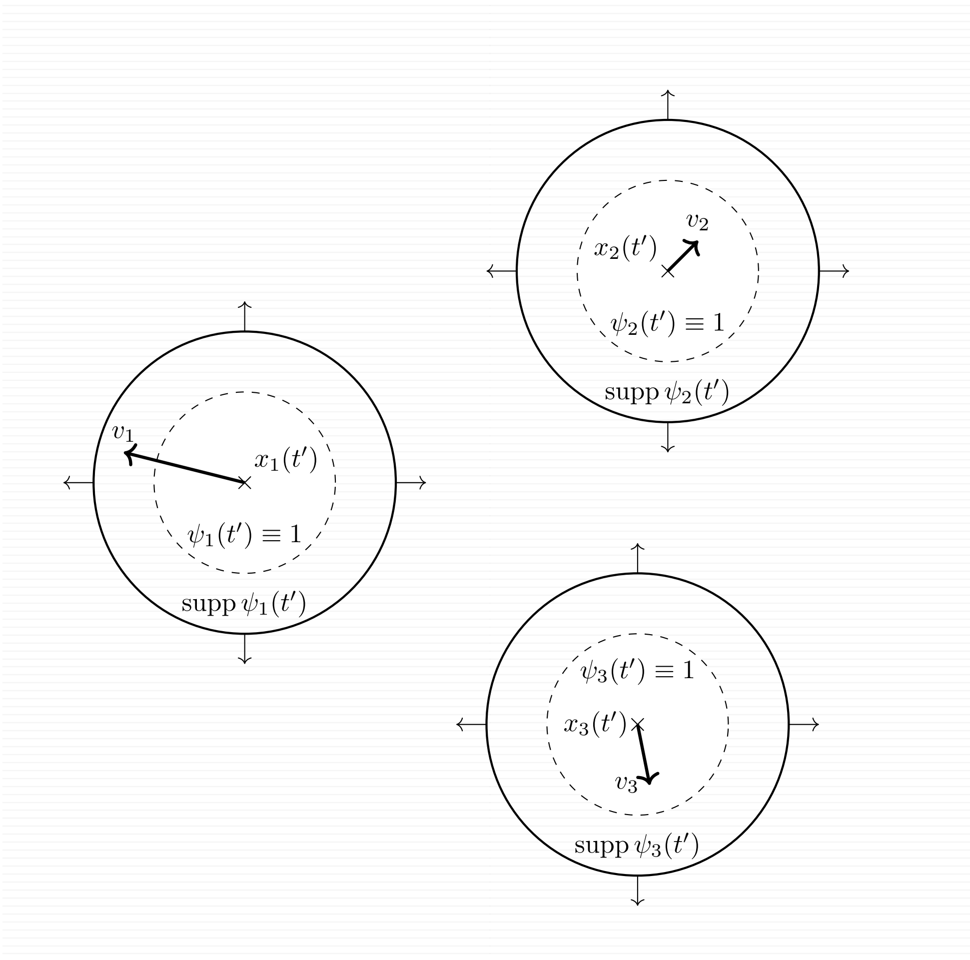

Figure 3.3.1 – Schematic representation for the partition of unity $(\psi_j(t'))$ (in dimension 2). The support of $\psi_0(t')$ is given by the horizontal lines.

Corollary 3.3.5. $\psi_0(t') \geq 0$ *for all* $t \geq 0$.

Proof. It readily follows from the facts that $0 \leq \psi_j \leq 1$ for each j, each ψ_j has disjoint support and

$$\psi_0 = 1 - \sum_j \psi_j. \qquad \square$$

With this definition, $(\psi_j(t'))_j$ is a (time-dependent) smooth partition of unity in space for all $t \geq 0$ (see Figure 3.3.1 for a schematic representation). We can now define *localized quantities* which will turn out to be almost conserved: for $j = 0, \dots, N$ and $t \geq 0$, set

$$M_j(t', v) \coloneqq \int |v|^2 \, \psi_j(t') \, \mathrm{d}x, \qquad \mathcal{J}_j(t', v) \coloneqq \operatorname{Im} \int \nabla v \, \overline{v} \, \psi_j(t') \, \mathrm{d}x,$$
$$E_j(t', v) \coloneqq \frac{1}{2} \int |\nabla v|^2 \, \psi_j(t') \, \mathrm{d}x - \lambda \int |v|^2 (\ln|v|^2 - 1) \, \psi_j(t') \, \mathrm{d}x.$$

We know that each G_j is well fitted for the action (for $j = 0$, take $G_0 = G^d$, $\omega_0 = 0$ and $v_0 = 0$)

$$S_j(v) \coloneqq E(v) + \Big(2\lambda\omega_j + \frac{|v_j|^2}{2}\Big) M(v) - v_j \cdot \mathcal{J}(v).$$

Then, we also define as well localized actions (for $j = 0, \dots, N$) by:

$$S_j^{\mathrm{loc}}(t', v) \coloneqq E_j(t', v) + \Big(2\lambda\omega_j + \frac{|v_j|^2}{2}\Big) M_j(t', v) - v_j \cdot \mathcal{J}_j(t', v).$$

Finally, we define a localized action-like functional for multi-solitons:

$$S^{\mathrm{loc}}(t',v) := \sum_j S_j^{\mathrm{loc}}(t',v) = E(v) + \sum_{j\geq 1}\Big(2\lambda\omega_j + \frac{|v_j|^2}{2}\Big)M_j(t',v) - v_j\cdot\mathcal{J}_j(t',v).$$

Our aim is to prove that $S^{\mathrm{loc}}(t',v)$ is almost coercive around $G(t')$, but also that $S^{\mathrm{loc}}(t',u_n(t'))$ slowly varies. To be more precise, we will prove the two following propositions.

Proposition 3.3.6 (Almost-coercivity)**.** *If t_* is large enough, then for all $t\in[t^\dagger,T_n-T'']$,*

$$S^{\mathrm{loc}}(t',u_n(t')) - \sum_{j\geq 1} S_j(G_j) \geq \frac{1}{2}\|w_n(t')\|_{\dot H^1}^2 - C_0\, t^{-1}e^{-\frac{\lambda(v_* t)^2}{2}}.$$

Proposition 3.3.7 (Slow variations)**.** *For all $t\in[t^\dagger,T_n-T'']$, there holds*

$$S^{\mathrm{loc}}(t',u_n(t')) - \sum_{j\geq 1} S_j(G_j) \leq C_0\, t^{-1}e^{-\frac{\lambda(v_* t)^2}{2}}.$$

Before proving these two propositions, we show how they lead to the Bootstrap Property 3.3.2.

Proof of Proposition 3.3.2. By Proposition 3.3.6, for all $t\in[t^\dagger,T_n-T'']$

$$\|w_n(t')\|_{\dot H^1}^2 \leq 2\Big(S^{\mathrm{loc}}(t',u_n(t')) - \sum_j S_j(G_j)\Big) + C_0\, t^{-1}e^{-\frac{\lambda(v_* t)^2}{2}}.$$

Moreover, Proposition 3.3.7 yields

$$S^{\mathrm{loc}}(t',u_n(t')) - \sum_j S_j(G_j) \leq C_0\, t^{-1}e^{-\frac{\lambda(v_* t)^2}{2}}.$$

Thus, combining this with the previous inequality, we get as soon as t_* is large enough:

$$\|w_n(t')\|_{\dot H^1}^2 \leq \frac{1}{4}e^{-\frac{\lambda(v_* t)^2}{2}}, \qquad \forall t\in[t^\dagger,T_n-T''],$$

which gives the conclusion. □

3.3.2. Almost-coercivity of the functional

We now prove Proposition 3.3.6, which requires to linearize the functional S^{loc} in terms of w_n and G. However, E is not $\mathcal{C}^2$ because of its potential energy, and we cannot linearize exactly like in [116, 53] for example. However, a weaker expansion still holds in this sense:

Lemma 3.3.8. *For all $t\geq 0$, $n\in\mathbb{N}$ and $j\in\{1,\dots,N\}$, there holds*

$$S_j^{\mathrm{loc}}(t',u_n(t')) \geq S_j^{\mathrm{loc}}(t',G_j(t')) + H_j(t',w_n^j(t')) - C_0\int \big|w_n^j(t')\big|^2\psi_j(t')\,\mathrm{d}x \\ - \int \mathrm{Re}\Big(\nabla G_j(t')\,\overline{w_n^j(t')}\Big)\cdot\nabla\psi_j(t')\,\mathrm{d}x + v_j\cdot\mathrm{Im}\int \overline{G_j(t')}\,w_n^j(t')\,\nabla\psi_j(t')\,\mathrm{d}x, \quad (3.3.7)$$

where $w_n^j := u_n - G_j$ and

$$H_j(t',w) := \frac{1}{2}\int|\nabla w|^2\psi_j(t')\,\mathrm{d}x - 2\lambda\int|w|^2\ln\Big(1+|w|\Big)\psi_j(t')\,\mathrm{d}x + \Big(2\lambda\omega_j + \frac{|v_j|^2}{2}\Big)M_j(t',w) - v_j\cdot\mathcal{J}_j(t',w). \quad (3.3.8)$$

This Lemma is a corollary of a kind of weak expansion of the function $F_1(z)=|z|\ln|z|^2$ (see Lemma 3.3.18). The second term in (3.3.8) is not what one would expect in order to be able to reproduce the proof of [116], since it is not the linearized potential energy. However, since we have already obtained the L^2 uniform estimates for w_n, it is still enough in order to prove Proposition 3.3.6. Before proving this lemma, we show how it is used to prove Proposition 3.3.6.

Proof of Proposition 3.3.6. First of all, we will need some results about the computation of some quantities about G_j outside its "physical support". Set $\nu := \lambda v_* > 0$.

Lemma 3.3.9. *For all $j \in \{1, \dots, N\}$ and $k \in \{0, \dots, N\}$, there holds when $t \to \infty$*

$$\|G_j(t')\|_{L^2((1-\psi_j(t'))\,\mathrm{d}x)} + \|G_j(t')\|_{L^2(\|D^3_{xxx}\psi_k(t')\|\,\mathrm{d}x)} + \|G_j(t')\|_{L^2(|\nabla\psi_k(t')|\,\mathrm{d}x)} = o\Big(e^{-\frac{\nu}{2}t} e^{-\frac{\lambda(v_* t)^2}{4}}\Big),$$
$$\|\nabla G_j(t')\|_{L^2((1-\psi_j(t'))\,\mathrm{d}x)} + \||\nabla\psi_k(t')|\,\nabla G_j(t')\|_{L^2} = o\Big(e^{-\frac{\nu}{2}t} e^{-\frac{\lambda(v_* t)^2}{4}}\Big),$$
$$\|G_j(t')\|_{L^{2+\frac{1}{d}}((1-\psi_j(t'))\,\mathrm{d}x)} = o\Big(e^{-\frac{\nu}{2}t} e^{-\frac{\lambda(v_* t)^2}{4}}\Big),$$
$$\Big\|G_j(t')\big|x - x_j^*(t')\big|^2\Big\|_{L^2((1-\psi_j(t'))\,\mathrm{d}x)} + \Big\|G_j(t')\big|x - x_j^*(t')\big|^3\Big\|_{L^2((1-\psi_j(t'))\,\mathrm{d}x)} = o\Big(e^{-\frac{\nu}{2}t} e^{-\frac{\lambda(v_* t)^2}{4}}\Big).$$

The lemma follows from the support properties of ψ_k and the Gaussian decay of G_j. We postpone its proof to 3.A. In particular, since $0 \le \psi_k \le 1 - \psi_j$ for all $k \ne j$, this gives a simple corollary:

Corollary 3.3.10. *For all $k \ge 0$, $j \ge 1$ such that $k \ne j$, there holds when $t \to \infty$*

$$\|G_j(t')\|_{L^2(\psi_k(t')\,\mathrm{d}x)} + \|G_j(t')\|_{L^{2+\frac{1}{d}}(\psi_k(t')\,\mathrm{d}x)} = o\Big(e^{-\frac{\nu}{2}t} e^{-\frac{\lambda(v_* t)^2}{4}}\Big),$$
$$\|\psi_k(t')\,\nabla G_j(t')\|_{L^2} + \|\nabla G_j(t')\|_{L^2(\psi_k(t')\,\mathrm{d}x)} = o\Big(e^{-\frac{\nu}{2}t} e^{-\frac{\lambda(v_* t)^2}{4}}\Big),$$
$$\Big\|G_j(t')\big|x - x_j^*(t')\big|^3\Big\|_{L^2(\psi_k(t')\,\mathrm{d}x)} = o\Big(e^{-\frac{\nu}{2}t} e^{-\frac{\lambda(v_* t)^2}{4}}\Big).$$

Moreover, we also know that $\ln|G_j(t')|^2 = 2\omega_j + d - 2\lambda|x - x_j^*(t')|^2$, so we can also derive the same decay estimate for $G_j(t')\ln|G_j(t')|^2$:

Corollary 3.3.11. *For all $k \ge 0$, $j \ge 1$ such that $k \ne j$, there holds when $t \to \infty$*

$$\Big\|G_j(t')\ln|G_j(t')|^2\Big\|_{L^2((1-\psi_j(t'))\,\mathrm{d}x)} = o\Big(e^{-\frac{\nu}{2}t} e^{-\frac{\lambda(v_* t)^2}{4}}\Big),$$
$$\Big\|G_j(t')\ln|G_j(t')|^2\Big\|_{L^2(\psi_k(t')\,\mathrm{d}x)} = o\Big(e^{-\frac{\nu}{2}t} e^{-\frac{\lambda(v_* t)^2}{4}}\Big).$$

Since the Gaussons get away from each other and with their Gaussian decay, we also show that they are almost orthogonal:

Lemma 3.3.12. *There exists $\alpha > 0$ such that for all $j \ne k \in \{1, \dots, N\}$ and $\ell \ge 1$, there holds*

$$\int |G_j(t')|\,|G_k(t')|(1 + |x - x_\ell^*(t')|^2)\,\mathrm{d}x + \int |\nabla G_j(t')|\,|G_k(t')|\,\mathrm{d}x + \int |\nabla G_j(t')|\,|\nabla G_k(t')|\,\mathrm{d}x = o\Big(e^{-\nu t} e^{-\frac{\lambda(v_* t)^2}{2}}\Big).$$

Proof. For the third term, we have

$$\begin{aligned}
\int |\nabla G_j(t')|\,|\nabla G_k(t')|\,\mathrm{d}x &= \int \big|-iv_j - 2\lambda(x - x_j^*(t'))\big|\big|-iv_k - 2\lambda(x - x_k^*(t'))\big||G_j(t',x)||G_k(t',x)|\,\mathrm{d}x \\
&= e^{d+\omega_j+\omega_k} \int \bigg|iv_j - 2\lambda\Big(y + \frac{x_k^*(t') - x_j^*(t')}{2}\Big)\bigg|\bigg|iv_k - 2\lambda\Big(y - \frac{x_k^*(t') - x_j^*(t')}{2}\Big)\bigg| \\
&\qquad\qquad \exp\Big[-2\lambda|y|^2 - \lambda\frac{(x_k^*(t') - x_j^*(t'))^2}{2}\Big]\,\mathrm{d}y \\
&\le C_0 \exp\Big[-\lambda\frac{\big|x_k^*(t') - x_j^*(t')\big|^2}{2}\Big] \int \Big(1 + |y|^2 + \big|x_k^*(t') - x_j^*(t')\big|^2\Big)\exp\Big[-2\lambda|y|^2\Big]\,\mathrm{d}y \\
&\le C_0 \exp\Big[-\lambda\frac{\big|x_k^*(t') - x_j^*(t')\big|^2}{2}\Big]\Big(\big|x_k^*(t') - x_j^*(t')\big|^2 + 1\Big) \\
&= o\Big(e^{-\nu t} e^{-\frac{\lambda(v_* t)^2}{2}}\Big),
\end{aligned}$$

as soon as τ (i.e. T'') is large enough, since $\big|x_k^*(t') - x_j^*(t')\big| \ge \varepsilon_0^{-1} + v_*(t + \tau)$ for all $j < k$ thanks to (3.2.5). The same kind of computation can be performed for the other terms. ☐

Now, we have all the tools that we need in order to prove Proposition 3.3.6 from Lemma 3.3.8. First, we show that the last three terms in (3.3.7) are negligible.

Corollary 3.3.13. *For all $n \in \mathbb{N}$, $t \in [t^\dagger, T_n - T'']$, $j \in \{1, \dots, N\}$,*

$$\left|\int \mathrm{Re}\Big(\nabla G_j(t')\,\overline{w_n^j(t')}\Big)\,\nabla\psi_j(t)\,\mathrm{d}x\right| + \left|\mathrm{Im}\int \overline{G_j(t')}\,w_n^j(t')\,\nabla\psi_j(t')\,\mathrm{d}x\right| + \int\left|w_n^j(t')\right|^2\psi_j(t')\,\mathrm{d}x \le C_0\,e^{-\nu t}e^{-\frac{\lambda(v_*t)^2}{2}}.$$

Proof. We recall that

$$w_n^j = w_n + \sum_{k\neq j} G_k,$$

so that

$$\left|\int \mathrm{Re}\Big(\nabla G_j(t')\,\overline{w_n^j(t')}\Big)\,\nabla\psi_j(t)\,\mathrm{d}x\right| \le \left|\int \mathrm{Re}\Big(\nabla G_j(t')\,\overline{w_n(t')}\Big)\,\nabla\psi_j(t)\,\mathrm{d}x\right| + \sum_{k\neq j}\left|\int \mathrm{Re}\Big(\nabla G_j(t')\,\overline{G_k(t')}\Big)\,\nabla\psi_j(t)\,\mathrm{d}x\right|.$$

By (3.3.5) and Lemma 3.3.9, there holds for all $t \in [t^\dagger, T_n - T'']$, as soon as τ is large enough,

$$\left|\int \mathrm{Re}\Big(\nabla G_j(t')\,\overline{w_n(t')}\Big)\,\nabla\psi_j(t)\,\mathrm{d}x\right| \le \|w_n(t')\|_{L^2}\|\nabla\psi_k(t')\,\nabla G_j(t')\|_{L^2} \le C_0\,e^{-\nu t}e^{-\frac{\lambda(v_*t)^2}{2}}.$$

Moreover, Lemma 3.3.12 gives

$$\left|\int \mathrm{Re}\Big(\nabla G_j(t')\,\overline{G_k(t')}\Big)\,\nabla\psi_j(t)\,\mathrm{d}x\right| \le \int|\nabla G_j(t')|\,|G_k(t')|\,\mathrm{d}x \le C_0\,e^{-\nu t}e^{-\frac{\lambda(v_*t)^2}{2}},$$

which gives the conclusion for the first term. A similar computation holds for the second term. As for the last term, as long as τ is large enough, an easy computation yields

$$\left\|w_n^j(t')\right\|_{L^2(\psi_j(t')\,\mathrm{d}x)} \le \|w_n(t')\|_{L^2} + \sum_{k\neq j}\|G_k(t')\|_{L^2(\psi_j(t')\,\mathrm{d}x)} \le C_0\,e^{-\frac{\nu}{2}t}e^{-\frac{\lambda(v_*t)^2}{4}}.$$ □

Then, we can also substitute $S_j^{\mathrm{loc}}(t', G_j(t'))$ by $S_j(G_j)$ up to a negligible term.

Corollary 3.3.14. *For all $j \in \{1, \dots, N\}$, for all $t \ge 0$*

$$\left|S_j^{\mathrm{loc}}(t', G_j(t')) - S_j(G_j)\right| \le C_0\,e^{-\nu t}e^{-\frac{\lambda(v_*t)^2}{2}}.$$

Proof. With a simple computation,

$$S_j(G_j) - S_j^{\mathrm{loc}}(t', G_j(t')) = \frac{1}{2}\int|\nabla G_j(t')|^2\,(1-\psi_j(t'))\,\mathrm{d}x - \lambda\int|G_j(t')|^2(\ln|G_j(t')|^2 - 1)\,(1-\psi_j(t'))\,\mathrm{d}x$$
$$+ \left(2\lambda\omega_j + \frac{|v_j|^2}{2}\right)\int|G_j|^2(1-\psi_j(t'))\,\mathrm{d}x - v_j\cdot\mathrm{Im}\int\nabla G_j\,\overline{G_j}\,(1-\psi_j(t'))\,\mathrm{d}x.$$

The conclusion readily follows from Lemma 3.3.9 and Corollary 3.3.11. □

An important feature of $H = \sum_j H_j$ is that it is coercive in H^1, up to an L^2 norm. Since we already know that the L^2 norm of w_n is negligible, its H^1 norm is therefore controlled by H. In order to prove this, we have this first result about some coercivity of H_j:

Lemma 3.3.15. *For all $n \in \mathbb{N}$, $t \in [t^\dagger, T_n - T'']$, $j \in \{1, \dots, N\}$,*

$$H_j(t', w_n^j(t')) \ge \frac{1}{2}\int|\nabla w_n|^2\,\psi_j(t')\,\mathrm{d}x - C_0\,e^{-\frac{\nu}{2}t}e^{-\frac{\lambda(v_*t)^2}{2}}.$$

Proof. From (3.3.8), we have

$$H_j(t', w_n^j(t')) \coloneqq \frac{1}{2}\int\left|\nabla w_n^j(t')\right|^2\psi_j(t')\,\mathrm{d}x - 2\lambda\int\left|w_n^j(t')\right|^2\,\ln\Big(1+\left|w_n^j(t')\right|\Big)\,\psi_j(t')\,\mathrm{d}x$$
$$+ \Big(2\lambda\omega_j + \frac{|v_j|^2}{2}\Big)M_j(t', w_n^j(t')) - v_j\cdot\mathcal{J}_j(t', w_n^j(t')).$$

We also recall that $w_n^j = w_n + \sum_{k\neq j} G_k$.

— For the first term, we have

$$\int \left|\nabla w_n^j(t')\right|^2 \psi_j(t')\,\mathrm{d}x = \int \left|\nabla w_n(t')\right|^2 \psi_j(t')\,\mathrm{d}x + 2\sum_{k\neq j} \operatorname{Re}\int \overline{\nabla w_n(t')}\,\nabla G_k(t')\,\psi_j(t')\,\mathrm{d}x + \int \left|\sum_{k\neq j}\nabla G_k(t')\right|^2 \psi_j(t')\,\mathrm{d}x.$$

For the second term in the right-hand side, we compute for any $k \neq j$ thanks to (3.3.4) and Corollary 3.3.10:

$$\left|\operatorname{Re}\int \overline{\nabla w_n(t')}\,\nabla G_k(t')\,\psi_j(t')\,\mathrm{d}x\right| \leq \|\nabla w_n(t')\|_{L^2} \|\psi_j(t')\,\nabla G_k(t')\|_{L^2} \leq C_0\, e^{-\frac{\nu}{2}t} e^{-\frac{\lambda(v_* t)^2}{2}}.$$

As for the third term, using Corollary 3.3.10, we compute in the same way:

$$\left\|\sum_{k\neq j}\nabla G_k(t')\right\|_{L^2(\psi_j(t')\,\mathrm{d}x)} \leq \sum_{k\neq j}\|\nabla G_k(t')\|_{L^2(\psi_j(t')\,\mathrm{d}x)} \leq C_0\, e^{-\frac{\nu}{2}t} e^{-\frac{\lambda(v_* t)^2}{4}}.$$

— For the second term, we use the fact that there exists $C_d > 0$ such that for all $z \in \mathbb{R}_+$,

$$z^2 \,\ln\Big(1+z\Big) \leq C_d\, z^{2+\frac{1}{d}},$$

so that

$$\left|\int \left|w_n^j(t')\right|^2 \,\ln\Big(1+\left|w_n^j(t')\right|\Big)\,\psi_j(t')\,\mathrm{d}x\right| \leq C_d \left\|w_n^j(t')\right\|^{2+\frac{1}{d}}_{L^{2+\frac{1}{d}}(\psi_j(t')\,\mathrm{d}x)}.$$

Then, using Corollary 3.3.10, we have

$$\begin{aligned}\left\|w_n^j(t')\right\|_{L^{2+\frac{1}{d}}(\psi_j(t')\,\mathrm{d}x)} &\leq \|w_n(t')\|_{L^{2+\frac{1}{d}}} + \sum_{k\neq j}\|G_k(t')\|_{L^{2+\frac{1}{d}}(\psi_j(t')\,\mathrm{d}x)} \\ &\leq C_d\,\|w_n(t')\|_{H^1} + C_0\, e^{-\frac{\nu}{2}t} e^{-\frac{\lambda(v_* t)^2}{4}} \leq C_0\, e^{-\frac{\lambda(v_* t)^2}{4}}.\end{aligned}$$

Thus,

$$\left|\int \left|w_n^j(t')\right|^2 \,\ln\Big(1+\left|w_n^j(t')\right|\Big)\,\psi_j(t')\,\mathrm{d}x\right| \leq C_0\, e^{-\frac{\nu}{2}t} e^{-\frac{\lambda(v_* t)^2}{2}}.$$

— The last two terms can be easily estimated in the same way since $M_j(t', w_n^j(t')) = \left\|w_n^j\right\|^2_{L^2(\psi_j(t')\,\mathrm{d}x)}$ and

$$\left|\mathcal{J}_j(t', w_n^j(t'))\right| \leq \left\|w_n^j\right\|_{L^2(\psi_j(t')\,\mathrm{d}x)} \left\|\nabla w_n^j\right\|_{L^2(\psi_j(t')\,\mathrm{d}x)}.$$

□

By putting Corollaries 3.3.13 and 3.3.14 and Lemma 3.3.15 in Lemma 3.3.8, we easily deduce a nice "coercivity" property for the localized functionals S_j^{loc} for $j \geq 1$:

Corollary 3.3.16. *For all $n \in \mathbb{N}$, $t \in [t^\dagger, T_n - T'']$, $j \in \{1, \ldots, N\}$,*

$$S_j^{\mathrm{loc}}(t', u_n(t')) - S_j(G_j) \geq \frac{1}{2}\int |\nabla w_n|^2\,\psi_j(t')\,\mathrm{d}x - C_0\, e^{-\frac{\nu}{2}t} e^{-\frac{\lambda(v_* t)^2}{2}}.$$

As for the case $j = 0$, a similar property holds:

Lemma 3.3.17. *For all $n \in \mathbb{N}$, $t \in [t^\dagger, T_n - T'']$*

$$S_0^{\mathrm{loc}}(t', u_n(t')) \geq \frac{1}{2}\int |\nabla w_n|^2\,\psi_0(t')\,\mathrm{d}x - C_0\, e^{-\frac{\nu}{2}t} e^{-\frac{\lambda(v_* t)^2}{2}}.$$

Proof.

$$S_0^{\mathrm{loc}}(t', u_n(t')) = \frac{1}{2}\int |\nabla u_n(t')|^2\,\psi_0(t')\,\mathrm{d}x - \lambda\int |u_n(t')|^2(\ln|u_n(t')|^2 - 1)\,\psi_0(t')\,\mathrm{d}x.$$

— For the first term:

$$\int |\nabla u_n|^2 \, \psi_0(t')\,\mathrm{d}x = \int |\nabla w_n|^2 \, \psi_0(t')\,\mathrm{d}x + 2\int \mathrm{Re}\,\nabla w_n \cdot \overline{\nabla G}\, \psi_0(t')\,\mathrm{d}x + \int |\nabla G|^2 \, \psi_0(t')\,\mathrm{d}x.$$

For the second term of the right-hand side, we have:

$$\begin{aligned}\left| \int \mathrm{Re}\,\nabla w_n \cdot \overline{\nabla G}\, \psi_0(t')\,\mathrm{d}x \right| &\leq \|\nabla w_n\|_{L^2} \|\psi_0(t')\, \nabla G(t')\|_{L^2} \\ &\leq \|\nabla w_n\|_{L^2} \sum_{j\geq 1} \|\psi_0(t')\, \nabla G_j(t')\|_{L^2} \\ &\leq C_0\, e^{-\frac{\nu}{2}t} e^{-\frac{\lambda(v_* t)^2}{2}}.\end{aligned}$$

Then we use Corollary 3.3.10 for the last term.

— For the second term, we show it is negligible by using the fact that for all $y > e$:

$$y\,(\ln y - 1) \leq C_d\, y^{1+\frac{1}{2d}}$$

for some $C_d > 0$. Thus,

$$\begin{aligned}\int |u_n|^2 (\ln|u_n|^2 - 1)\, \psi_0(t)\,\mathrm{d}x &\leq \int_{|u_n|^2 > e} |u_n|^2 (\ln|u_n|^2 - 1)\, \psi_0(t)\,\mathrm{d}x \\ &\leq C_d\, \|u_n(t')\|^{2+\frac{1}{d}}_{L^{2+\frac{1}{d}}(\psi_0(t')\,\mathrm{d}x)}.\end{aligned}$$

Then, the conclusion comes from:

$$\begin{aligned}\|u_n(t')\|_{L^{2+\frac{1}{d}}(\psi_0(t')\,\mathrm{d}x)} &\leq \|w_n(t')\|_{L^{2+\frac{1}{d}}} + \sum_{j\geq 1} \|G_j(t')\|_{L^{2+\frac{1}{d}}(\psi_0(t')\,\mathrm{d}x)} \\ &\leq C_d\, \|w_n(t')\|_{H^1} + C_0\, e^{-\frac{\nu}{2}t} e^{-\frac{\lambda(v_* t)^2}{4}} \leq C_0\, e^{-\frac{\lambda(v_* t)^2}{4}}.\end{aligned}$$ □

Proposition 3.3.6 is then a simple corollary from these results, by summing over j Corollary 3.3.16 and Lemma 3.3.17.

Proof of Lemma 3.3.8. This lemma mostly relies on an inequality for the potential energy with an expression near the expected expansion. To prove this, set

$$\begin{aligned} F_1 : \mathbb{C} &\to \mathbb{R} \\ z &\mapsto |z|^2\, (\ln|z|^2 - 1),\end{aligned}$$

so that the potential energy is $-\lambda \int F_1(v)$. Then, the following inequality holds:

Lemma 3.3.18. *For all $z_1, z_2 \in \mathbb{C}$, set $\zeta := z_1 - z_2$. Then*

$$F_1(z_1) \leq F_1(z_2) + 2\,\mathrm{Re}\left(z_2\overline{\zeta}\right)\, \ln|z_2|^2 + 2\,|\zeta|^2 \left(\ln\left(\max(|z_2|, |z_1|) \right) + 1 \right).$$

Remark 3.3.19. In the case $z_2 = 0$, the second term of the right-hand side is to be understood as being 0. If furthermore $z_1 = 0$, then so is the last term of the right-hand side.

Remark 3.3.20. Like already pointed out, the third term of the right-hand side is not what one would expect in order to be able to reproduce the proof of [116, 53] for instance. Indeed, the expected formula would be something of the form:

$$F_1(z_1) = F_1(z_2) + 2\,\mathrm{Re}\left(z_2\overline{\zeta}\right)\, \ln|z_2|^2 + |\zeta|^2 \ln|z_2|^2 + 2\frac{1}{|z_2|^2}\left(\mathrm{Re}\left(z_2\overline{\zeta}\right)\right)^2 + h(\zeta, z_2),$$

where h is (at least) bounded when ζ and z_2 are bounded (and presumably of order more than 2 in ζ). However, taking $z_1 = 1$ and $z_2 \to 0$ gives a simple counter-example for such an expansion.

Moreover, if one takes $z_1 = u(x)$ for some $u \in W$ and $z_2 = e^{-|x|^2}$ for instance and integrate, it gives:

$$\int F_1(u(x)) = \int F_1(e^{-|x|^2}) - 4\int \mathrm{Re}\left(e^{-|x|^2}\overline{\zeta(x)}\right)\, |x|^2 - 2\int |\zeta(x)|^2 |x|^2 + 2\int \left(\mathrm{Re}\left(\zeta(x)\right)\right)^2 + \int h(\zeta(x), e^{-|x|^2}),$$

where one would want the last term to be controlled by the W-norm of $\zeta(x)$. With so, every term is bounded expect the third term of the right-hand side, which could be $-\infty$ if we take $u \notin \mathcal{F}(H^1)$ (such a u exists).

Proof. For this proof only, we use the identification $\mathbb{C} \approx \mathbb{R}^2$, and we see F_1 as a function from $\mathbb{R}^2$ into $\mathbb{R}$:

$$\begin{aligned} F_1 : \mathbb{C} \approx \mathbb{R}^2 &\to \mathbb{R} \\ z = \begin{bmatrix} z_r \\ z_i \end{bmatrix} \in \mathbb{R}^2 &\mapsto |z|^2 \left(\ln|z|^2 - 1\right). \end{aligned}$$

For $z = 0$, $F_1(0) = 0$. Then, F is differentiable on $\mathbb{C}$ and twice differentiable on $\mathbb{C} \setminus \{0\}$ and we can compute for $z \neq 0$:

$$\nabla F_1(z) = \begin{bmatrix} 2z_r \ln|z|^2 \\ 2z_i \ln|z|^2 \end{bmatrix} = 2z \ln|z|^2.$$

We also compute $\nabla F(0) = 0$. Then, for all $z \neq 0$, we can differentiate again:

$$D^2 F_1(z) = 2 \begin{bmatrix} \ln|z|^2 + 2\frac{z_r^2}{|z|^2} & 2\frac{z_r z_i}{|z|^2} \\ 2\frac{z_r z_i}{|z|^2} & \ln|z|^2 + 2\frac{z_i^2}{|z|^2} \end{bmatrix} = 2\, R_z^{-1} L(z) R_z,$$

where R_z is the rotation which maps z onto the real positive half-line of $\mathbb{C}$ and $L(z)$ is defined by:

$$L(z) = \begin{bmatrix} \ln|z|^2 + 2 & 0 \\ 0 & \ln|z|^2 \end{bmatrix}.$$

In particular, we see that for all $z \neq 0$ and all $h \in \mathbb{R}^2$, there holds

$$\langle h, D^2 F_1(z)\, h\rangle = 2\langle R_z h, L(z) R_z h\rangle \leq 2(\ln|z|^2 + 2)\, |R_z h|^2 = 4(\ln|z| + 1)\, |h|^2.$$

Take $z_1, z_2 \in \mathbb{C}$ and set $\zeta := z_1 - z_2$. In the case $z_1 = z_2 = 0$, Remark 3.3.19 makes the inequality trivial to prove. Otherwise, Taylor's formula with integral form gives:

$$F_1(z_1) = F_1(z_2) + \langle \zeta, \nabla F_1(z_2)\rangle + \int_0^1 \langle \zeta, D^2 F_1(z_2 + t\zeta)\, \zeta\rangle\, (1-t)\, \mathrm{d}t. \tag{3.3.9}$$

The second term of the right-hand side is exactly what we expect since:

$$\langle \zeta, \nabla F_1(z_2)\rangle = \langle \zeta, 2z_2 \ln|z_2|^2\rangle = 2\, \langle \zeta, z_2\rangle\, \ln|z_2|^2 = 2 \operatorname{Re}\left(z_2 \overline{\zeta}\right)\, \ln|z_2|^2,$$

and is 0 if $z_2 = 0$. As for the last term of the right-hand side, it can be estimated as previously:

$$\int_0^1 \langle \zeta, D^2 F_1(z_2 + tz)\, z\rangle\, (1-t)\, \mathrm{d}t \leq 4 \int_0^1 (\ln|z_2 + t\zeta| + 1)\, |\zeta|^2\, (1-t)\, \mathrm{d}t.$$

Moreover, there holds for all $t \in [0,1]$

$$|z_2 + t\zeta| = |(1-t)z_2 + tz_1| \leq (1-t)|z_2| + t|z_1| \leq \max(|z_2|, |z_1|)$$

Therefore, since $\ln$ is increasing,

$$\begin{aligned} \int_0^1 \langle \zeta, D^2 F_1(z_2 + t\zeta)\, \zeta\rangle\, (1-t)\, \mathrm{d}t &\leq 4\, |\zeta|^2 \int_0^1 \Big(\ln\Big(\max(|z_2|, |z_1|)\Big) + 1\Big)\, (1-t)\, \mathrm{d}t \\ &\leq 2\, |\zeta|^2 \Big(\ln\Big(\max(|z_2|, |z_1|)\Big) + 1\Big) \end{aligned}$$

The conclusion readily follows from putting this inequality into (3.3.9). □

Corollary 3.3.21. *For all $x \in \mathbb{R}$, $t \geq 0$, $n \in \mathbb{N}$ and $j \in \{1, \dots, N\}$, there holds*

$$F_1(u_n(t', x)) \leq F_1(G_j(t', x)) + 2 \operatorname{Re}\Big(G_j(t', x)\, \overline{w_n^j(t', x)}\Big)\, \ln|G_j(t', x)|^2 + 2\left|w_n^j(t', x)\right|^2 \Big(\ln\Big(1 + \left|w_n^j(t', x)\right|\Big) + C_0\Big),$$

where $w_n^j := u_n - G_j$.

Proof. Applying Lemma 3.3.18 with $z_1 = u_n(t', x)$ and $z_2 = G_j(t', x)$ gives

$$F_1(u_n(t',x)) \le F_1(G_j(t',x)) + 2\,\mathrm{Re}\left(G_j(t',x)\,\overline{w_n^j(t',x)}\right)\,\ln|G_j(t',x)|^2 + \left|w_n^j(t',x)\right|^2\left(\ln\left(\max(|G_j(t',x)|,|u_n(t',x)|)\right)+1\right), \quad (3.3.10)$$

We know that

$$|u_n(t',x)| \le |G_j(t',x)| + \left|w_n^j(t',x)\right|$$

and

$$\|G_j(t')\|_{L^\infty} \le C_0.$$

Thus,

$$\max(|G_j(t',x)|,|u_n(t',x)|) \le C_0 + \left|w_n^j(t',x)\right|,$$

which yields

$$\ln\left(\max(|G_j(t',x)|,|u_n(t',x)|)\right) \le \ln\left(C_0 + \left|w_n^j(t',x)\right|\right) \le C_0 + \ln\left(1 + \left|w_n^j(t',x)\right|\right).$$

The result follows by putting this estimate into (3.3.10). □

The proof of Lemma 3.3.8 readily follows from expanding $S_j^{\mathrm{loc}}(t', u_n(t'))$ in terms of $w_n^j(t')$, using Corollary 3.3.21 for the "expansion" of the localized potential energy.

3.3.3. Slow variations of the functional

In this subsection, we prove Proposition 3.3.7. Again, this proposition is similar to that in [116, 53] for example, and the proof is almost the same. However, some minor changes occur. The first one comes from the fact that we took a "true" d-dimensional partition of unity, and thus the link between the time and space derivatives of this partition is less obvious, yet relatively similar:

Lemma 3.3.22. *For all $j \ge 1$, $t \ge 0$ and $x \in \mathbb{R}$, there holds*

$$|\partial_t \psi_j(t',x)| \le C_0\,|\nabla\psi_j(t',x)|.$$

Thanks to this link, it is easy to prove that S^{loc} slowly varies in the same way as in [116, 53]:

Lemma 3.3.23. *For all $t \in [t^\dagger, T_n - T'']$, there holds*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} S^{\mathrm{loc}}(t', u_n(t'))\right| \le C_0\, e^{-\frac{\lambda(v_* t)^2}{4}}.$$

Proof. We already know that the energy $E(u_n(t))$ is conserved. To estimate the variations of $S(t', u_n(t'))$, we only have to study the variations of the localized masses $M_j(t', u_n(t'))$ and momenta $\mathcal{J}_j(t', u_n(t'))$. Thanks to the expression of the partition of unity, we only need to compute (for $j \ge 1$ only since $\omega_0 = 0$ and $v_0 = 0$)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int |u_n|^2\,\psi_j\,\mathrm{d}x &= \int \mathrm{Im}\left(\Delta u_n \overline{u_n}\right)\psi_j\,\mathrm{d}x + \int |u_n|^2\,\partial_t\psi_j\,\mathrm{d}x\\ &= \int \mathrm{Im}\left(\nabla u_n \overline{u_n}\right)\cdot\nabla\psi_j\,\mathrm{d}x + \int |u_n|^2\,\partial_t\psi_j\,\mathrm{d}x\\ \left|\frac{\mathrm{d}}{\mathrm{d}t}\int |u_n|^2\,\psi_j\,\mathrm{d}x\right| &\le C_0\int \left(|\nabla u_n|^2 + |u_n|^2\right)|\nabla\psi_j|\,\mathrm{d}x.\end{aligned}$$

Similarly, we have for $\mathcal{J}_j(t', u_n(t'))$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{Im}\left(\nabla u_n(t')\,\overline{u_n(t')}\right)\psi_j(t')\,\mathrm{d}x = \int \mathrm{Re}\left(\left(\overline{\nabla u_n}\cdot\nabla\psi_j(t')\right)\nabla u_n\right)\mathrm{d}x - \int \mathrm{Im}\left(\nabla u_n(t')\,\overline{u_n(t')}\right)\partial_t\psi_j(t')\,\mathrm{d}x - \lambda\int |u_n(t')|^2\,\nabla\psi_j(t')\,\mathrm{d}x - \frac{1}{4}\int |u_n(t')|^2\,\nabla\cdot D^2_{xx}\psi_j(t')\,\mathrm{d}x.$$

Therefore there holds again

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_j(t',u_n(t'))\right| \le C_0\int \left(|\nabla u_n(t')|^2 + |u_n(t')|^2\right)|\nabla\psi_j(t')|\,\mathrm{d}x + C_0\int |u_n(t')|^2\left\|D^3_{xxx}\psi_j(t')\right\|\mathrm{d}x.$$

Now, remark that

$$\int \Big(|\nabla u_n(t')|^2 + |u_n(t')|^2\Big) |\nabla \psi_j(t')| \,\mathrm{d}x \leq 2\bigg(\int \Big(|\nabla G(t')|^2 + |G(t',x)|^2\Big) |\nabla \psi_j(t')| \,\mathrm{d}x + C_0 \left\|w_n(t')\right\|_{H^1(\mathbb{R})}^2\bigg)$$

and

$$\int |u_n(t',x)|^2 \left\|D^3_{xxx}\psi_j(t')\right\| \mathrm{d}x \leq 2\bigg(\int |G(t',x)|^2 \left\|D^3_{xxx}\psi_j(t')\right\| \mathrm{d}x + C_0 \left\|w_n(t')\right\|_{L^2(\mathbb{R})}^2\bigg).$$

By assumption, we know that for all $t \in [t^\dagger, T_n - T'']$

$$\left\|w_n(t')\right\|_{H^1(\mathbb{R})}^2 \leq 2e^{-\frac{\lambda(v_* t)^2}{2}}.$$

Moreover, by Lemma 3.3.9, we have

$$\int \Big(|\nabla G(t',x)|^2 + |G(t',x)|^2\Big) |\nabla \psi_j(t')| \,\mathrm{d}x = o\Big(e^{-\nu t} e^{-\frac{\lambda(v_* t)^2}{2}}\Big),$$

$$\int |G(t',x)|^2 \left\|D^3_{xxx}\psi_j(t')\right\| \mathrm{d}x = o\Big(e^{-\nu t} e^{-\frac{\lambda(v_* t)^2}{2}}\Big).$$

Consequently,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int |u_n(t')|^2 \psi_j(t') \,\mathrm{d}x\right| + \left|\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{Im}\Big(\nabla u_n(t')\, \overline{u_n(t')}\Big)\psi_j(t') \,\mathrm{d}x\right| \leq C_0\, e^{-\frac{\lambda(v_* t)^2}{2}}$$

Plugging this estimate into the expression of M_j and $\mathcal{J}_j$ gives for all $t \geq t^\dagger$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} M_j(t', u(t'))\right| + \left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}_j(t', u(t'))\right| \leq C_0\, e^{-\frac{\lambda(v_* t)^2}{2}},$$

and the conclusion readily follows. □

The fact that the convergence is Gaussian instead of exponential gives a free t^{-1} factor when integrating, which is enough to be negligible.

Corollary 3.3.24. *There holds for n large enough and $t \in [t^\dagger, T_n - T'']$:*

$$\left|S^{\mathrm{loc}}(t', u_n(t')) - S^{\mathrm{loc}}(T_n, G(T_n))\right| \leq C_0\, t^{-1} e^{-\frac{\lambda(v_* t)^2}{2}}.$$

Proof. Defining $\tilde{S}_n(s) \coloneqq S(s, u_n(s))$, we can estimate this difference thanks to the previous estimate:

$$S(t', u_n(t')) - S(T_n, G(T_n)) = -\int_{t'}^{T_n} \frac{\mathrm{d}\tilde{S}_n}{\mathrm{d}t}(s)\,\mathrm{d}s = -\int_t^{T_n - T''} \frac{\mathrm{d}\tilde{S}_n}{\mathrm{d}t}(T'' + s)\,\mathrm{d}s$$

$$|S(t', u_n(t')) - S(T_n, G(T_n))| \leq \int_t^{T_n - T''} C_0\, e^{-\frac{\lambda(v_* s)^2}{2}}\,\mathrm{d}s$$

$$\leq C_0\, t^{-1} e^{-\frac{\lambda(v_* t)^2}{2}},$$

thanks to Lemma 3.2.9 with $d = 1$ and $n = 0$. □

In the previous result, we have $S^{\mathrm{loc}}(T_n, G(T_n))$: we would like to have $S_j(G_j)$ instead. Thanks to Corollary 3.3.14, we only need $S^{\mathrm{loc}}(t', G_j(t'))$.

Lemma 3.3.25. *There holds for all $t \geq 0$ and $j \in \{1, \ldots, N\}$,*

$$\left|S_j^{\mathrm{loc}}(t', G(t')) - S_j^{\mathrm{loc}}(t', G_j(t'))\right| \leq C_0\, e^{-\nu t} e^{-\frac{\lambda(v_* t)^2}{2}},$$

$$\left|S_0^{\mathrm{loc}}(t', G(t'))\right| \leq C_0\, e^{-\nu t} e^{-\frac{\lambda(v_* t)^2}{2}}.$$

Proof. Decomposing $S_j^{\mathrm{loc}}(t', G(t'))$ and $S_j^{\mathrm{loc}}(t', G_j(t'))$, we get

$$\begin{aligned} S_j^{\mathrm{loc}}(t', G(t')) - S_j^{\mathrm{loc}}(t', G_j(t')) = \frac{1}{2}\Big(&\int |\nabla G(t')|^2 \psi_j(t')\,\mathrm{d}x - \int |\nabla G_j(t')|^2 \psi_j(t')\,\mathrm{d}x\Big) \\ &- \lambda\bigg(\int |G(t')|^2 \ln|G(t')|^2 \psi_j(t')\,\mathrm{d}x - \int |G_j(t')|^2 \ln|G_j(t')|^2 \psi_j(t')\,\mathrm{d}x\bigg) \\ &+ \Big(2\lambda\omega_j + \lambda + \frac{|v_j|^2}{2}\Big)(M_j(t', G(t')) - M_j(t', G_j(t'))) \\ &- v_j \cdot (\mathcal{J}_j(t', G(t')) - \mathcal{J}_j(t', G_j(t'))). \end{aligned}$$

— For the first term, decomposing $G = \sum_j G_j$,

$$\int |\nabla G(t')|^2 \psi_j(t')\,\mathrm{d}x - \int |\nabla G_j(t')|^2 \psi_j(t')\,\mathrm{d}x = \sum_{(k,\ell)\neq(j,j)} \operatorname{Re}\int \nabla G_\ell(t')\cdot\overline{\nabla G_k(t')}\,\psi_j(t')\,\mathrm{d}x$$
$$= \sum_{k\neq\ell}\operatorname{Re}\int \nabla G_\ell(t')\cdot\overline{\nabla G_k(t')}\,\psi_j(t')\,\mathrm{d}x + \sum_{k\neq j}\int |\nabla G_k(t')|^2\,\psi_j(t')\,\mathrm{d}x,$$

and the conclusion with Lemma 3.3.9 and Lemma 3.3.12.

— For the third and fourth terms, the same kind of decomposition can be used:

$$\mathcal{J}_j(t', G(t')) - \mathcal{J}_j(t', G_j) = \sum_{(k,\ell)\neq(j,j)} \operatorname{Im}\int \nabla G_k(t',x)\,\overline{G_l(t',x)}\,\psi_j(t',x)\,\mathrm{d}x,$$
$$M_j(t', G(t')) - M_j(t', G_j) = \sum_{(k,\ell)\neq(j,j)} \operatorname{Re}\int G_k(t',x)\,\overline{G_l(t',x)}\,\psi_j(t',x)\,\mathrm{d}x.$$

The conclusion comes in the same way.

— For the second term, we use a similar computation as in [69, Lemma 3.2]. Precisely, we will use the following lemma:

Lemma 3.3.26 ([69, Lemma 3.3])**.** *Set $F(z) \coloneqq z\ln|z|$. For all $z, \tilde z \in \mathbb{C}$ such that $|z| \leq 1$, $|\tilde z| \leq 1$ and $z \neq 0$, there holds*

$$|F(\tilde z) - F(z)| \leq |z - \tilde z|\Big[3 - \ln|z|\Big].$$

Then, by changing G_j and G into $\tilde G_j \coloneqq N^{-1}e^{-\omega}\,G_j$ and $\tilde G \coloneqq N^{-1}e^{-\omega}\,G$ respectively (with $\omega \coloneqq \max_k \omega_k$) and for all j, we get

$$\tilde G = \sum_k \tilde G_k, \qquad \sum_k \Big|\tilde G_k\Big| \leq 1.$$

Thus, for any $j \in \{1,\dots,N\}$, all $t \geq T$ and all $x \in \mathbb{R}$, there holds

$$\Big||G(t',x)|^2\ln|G(t',x)|^2 - |G_j(t',x)|^2\ln|G_j(t',x)|^2\Big|$$
$$= N^2e^{2\omega}\bigg|\Big|\tilde G(t',x)\Big|^2\Big(2\omega + \ln N^2 + \ln\Big|\tilde G(t',x)\Big|^2\Big) - \Big|\tilde G_j(t',x)\Big|^2\Big(2\omega + \ln N^2 + \ln\Big|\tilde G_j(t',x)\Big|^2\Big)\bigg|$$
$$\leq N^2e^{2\omega}\bigg|\Big|\tilde G(t',x)\Big|^2 - \Big|\tilde G_j(t',x)\Big|^2\bigg|\Big[2|\omega| + \ln N^2 + 3 - \ln\Big|\tilde G_j(t',x)\Big|^2\Big]$$
$$\leq \Big||G(t',x)|^2 - |G_j(t',x)|^2\Big|\Big[2|\omega| + \ln N^2 + 3 - \ln\Big|\tilde G_j(t',x)\Big|^2\Big]$$
$$\leq C_0\,|G(t',x) - G_j(t',x)|\Big(|G_j(t',x)| + \sum_k |G_k(t',x)|\Big)\Big[1 + \big|x - x_j^*(t')\big|^2\Big]$$
$$\leq C_0 \sum_{\ell\neq j}\sum_k |G_\ell(t',x)||G_k(t',x)|(1 + \big|x - x_j^*(t')\big|^2).$$

Multiplying by $\psi_j(t')$ and integrating over $\mathbb{R}^d$, we get

$$\left|\int |G(t')|^2\ln|G(t')|^2\,\psi_j(t')\,\mathrm{d}x - \sum_j \int |G_j(t')|^2\ln|G_j(t')|^2\,\psi_j(t')\,\mathrm{d}x\right|$$
$$\leq C_0\sum_{\ell\neq j}\sum_k \int |G_\ell(t')||G_k(t')|(1 + \big|x - x_j^*(t')\big|^2)\,\psi_j(t',x)\,\mathrm{d}x.$$

Thus, Lemma 3.3.9 and Lemma 3.3.12 yield the conclusion for this term.

The conclusion readily follows for the first inequality. As for the second one, the same computations can be done for $S_0^{\mathrm{loc}}(t', G(t')) - S_0^{\mathrm{loc}}(t', G_k(t'))$ for some k, so that

$$\Big|S_0^{\mathrm{loc}}(t', G(t')) - S_0^{\mathrm{loc}}(t', G_k(t'))\Big| \leq C_0\,e^{-\nu t}e^{-\frac{\lambda(v_* t)^2}{2}}$$

Moreover, it can easily be proved (thanks to Lemma 3.3.9) that

$$\left|S_0^{\text{loc}}(t', G_k(t'))\right| \leq C_0\, e^{-\nu t} e^{-\frac{\lambda(v_* t)^2}{2}},$$

and therefore the conclusion. □

We now have all the results we need to prove Proposition 3.3.7.

Proof of Proposition 3.3.7. We decompose the left-hand side in order to be able to apply the previous results:

$$\begin{aligned} S^{\text{loc}}(t', u_n(t')) - \sum_j S_j(G_j) = \Big(S^{\text{loc}}(t', u_n(t')) - S^{\text{loc}}(T_n, G(T_n))\Big) \\ + S_0^{\text{loc}}(T_n, G(T_n)) + \sum_{j\geq 1} \Big(S_j^{\text{loc}}(T_n, G(T_n)) - S_j^{\text{loc}}(T_n, G_j(T_n))\Big) \\ + \sum_{j\geq 1} \Big(S_j^{\text{loc}}(T_n, G_j(T_n)) - S_j(G_j)\Big). \end{aligned}$$

Thanks to Corollaries 3.3.24 (for the first line of the right-hand side) and 3.3.14 (for the last one) and to Lemma 3.3.25 (for the second one) along with the fact that

$$e^{-\nu(T_n - T'')} e^{-\frac{\lambda(v_*(T_n - T''))^2}{2}} \leq e^{-\nu t} e^{-\frac{\lambda(v_* t)^2}{2}}, \qquad \text{for all } 0 < t \leq T_n - T''$$

as soon as $T_n \geq T''$, we get

$$S^{\text{loc}}(t', u_n(t')) - \sum_j S(G_j) \leq C_0\, t^{-1} e^{-\frac{\lambda(v_* t)^2}{2}}.$$ □

3.4. Uniform $\mathcal{F}(H^1)$-estimates

The final step of our proof is the uniform estimates in $\mathcal{F}(H^1)$. The proof relies on an improvement of the computation of Section 3.2, using the uniform estimates in H^1 which are now proved.

Proposition 3.4.1. *For all $n \in \mathbb{N}$ such that $T_n \geq T$ and for all $t \in [0, T_n - T]$ (with $\check{t} := T + t$), there holds*

$$\left\|w_n(\check{t})\right\|_{\mathcal{F}(\dot{H}^1)} \leq C_0\, e^{-\frac{\lambda(v_* t)^2}{4}}.$$

Proof. We recall that $w_n = u_n - G$ satisfies

$$i\partial_t w_n + \frac{1}{2}\Delta w_n = -\lambda\Big[u_n \ln|u_n|^2 - \sum_k G_k \ln|G_k|^2\Big], \qquad w_n(T_n) = 0,$$

and there now holds for all $t \in [0, T_n - T]$

$$\left\|w_n(\check{t})\right\|_{L^2} \leq e^{-\frac{\lambda(v_* t)^2}{4}}, \qquad \left\|w_n(\check{t})\right\|_{\dot{H}^1} \leq e^{-\frac{\lambda(v_* t)^2}{4}}.$$

We also recall that we took T large enough so that for all $j \in \{1, \dots, N-1\}$ and $t \geq 1$, we have

$$\left|x_{j+1} - x_j + (v_{j+1} - v_j)\check{t}\right| \geq \varepsilon_0^{-1} + v_*(t + \tau),$$

as a consequence of (3.2.5). We compute the variations of this quantity:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int |x|^2 \left|w_n(\check{t})\right|^2 \mathrm{d}x = -\int |x|^2 \operatorname{Im}\Big[\Delta w_n(\check{t}) \overline{w_n(\check{t})}\Big] \mathrm{d}x \\ - 2\lambda \int |x|^2 \operatorname{Im}\Big[\Big[u_n(\check{t}) \ln\left|u_n(\check{t})\right|^2 - \sum_k G_k(\check{t}) \ln\left|G_k(\check{t})\right|^2\Big] \overline{w_n(\check{t})}\Big] \mathrm{d}x. \end{aligned} \tag{3.4.1}$$

— For the first term, performing an integration by parts, there holds

$$-\int |x|^2 \operatorname{Im}\Big[\Delta w_n(\check{t})\overline{w_n(\check{t})}\Big]\,\mathrm{d}x = 2\int x\cdot \operatorname{Im}\Big[\nabla w_n(\check{t})\overline{w_n(\check{t})}\Big]\,\mathrm{d}x.$$

This is easy to estimate with a Cauchy-Schwarz inequality:

$$\begin{aligned}\left|\int x\cdot \operatorname{Im}\Big[\nabla w_n(\check{t})\overline{w_n(\check{t})}\Big]\,\mathrm{d}x\right| &\le \left(\int \big|\nabla w_n(\check{t})\big|^2\,\mathrm{d}y\right)^{\frac12}\left(\int |x|^2\big|w_n(\check{t})\big|^2\,\mathrm{d}x\right)^{\frac12}\\ &\le e^{-\frac{\lambda(v_* t)^2}{4}}\left(\int |x|^2\big|w_n(\check{t})\big|^2\,\mathrm{d}x\right)^{\frac12}.\end{aligned}$$

Thus, we have

$$\left|\int |x|^2 \operatorname{Im}\Big[\Delta w_n(\check{t})\overline{w_n(\check{t})}\Big]\,\mathrm{d}x\right| \le C_0\, e^{-\frac{\lambda(v_* t)^2}{4}}\left(\int |x|^2\big|w_n(\check{t})\big|^2\,\mathrm{d}x\right)^{\frac12}. \tag{3.4.2}$$

— For the last term, we use again Lemma 3.2.7 and the fact that $w_n = u_n - G$, so that

$$\begin{aligned}&\left|\operatorname{Im}\left[\Big[u_n \ln|u_n|^2 - \sum_k G_k \ln|G_k|^2\Big]\overline{w_n}\right]\right|\\ &\qquad\le \left|\operatorname{Im}\left[\Big[u_n \ln|u_n|^2 - G\ln|G|^2\Big]\overline{w_n}\right]\right| + \left|\operatorname{Im}\left[\Big[G \ln|G|^2 - \sum_k G_k \ln|G_k|^2\Big]\overline{w_n}\right]\right|\\ &\qquad\le 2|w_n|^2 + \left|G\ln|G|^2 - \sum_k G_k\ln|G_k|^2\right||w_n|.\end{aligned}$$

Thus, we get

$$\begin{aligned}\left|\int |x|^2 \operatorname{Im}\left[\Big[u_n \ln|u_n|^2 - \sum_k G_k \ln|G_k|^2\Big]\overline{w_n}\right]\mathrm{d}x\right| &\le 2\int |x|^2\big|w_n(\check{t})\big|^2\,\mathrm{d}x\\ &\quad+\int |x|^2\left|G\ln|G|^2 - \sum_k G_k\ln|G_k|^2\right|\big|w_n(\check{t})\big|\,\mathrm{d}x\end{aligned}$$

For the second term of the right-hand side, performing a Cauchy-Schwarz inequality leads to

$$\int |x|^2\left|G\ln|G|^2 - \sum_k G_k\ln|G_k|^2\right|\big|w_n(\check{t})\big|\,\mathrm{d}x \le \left(\int |x|^2\left|G\ln|G|^2 - \sum_k G_k\ln|G_k|^2\right|^2\mathrm{d}x\right)^{\frac12}\left(\int |x|^2\big|w_n(\check{t})\big|^2\,\mathrm{d}x\right)^{\frac12}$$

For the first factor, we use the following result whose proof is postponed to Appendix 3.B.

Proposition 3.4.2. *For all $t \ge 0$, there holds*

$$\left\| |x|\left|G\ln|G|^2 - \sum_k G_k\ln|G_k|^2\right|\right\|_{L^2} \le C_0\, e^{-\frac{\lambda(v_* t)^2}{4}}$$

Thus we have

$$\left|\int |x|^2 \operatorname{Im}\left[\Big[u_n \ln|u_n|^2 - \sum_k G_k \ln|G_k|^2\Big]\overline{w_n}\right]\mathrm{d}x\right| \le 2\int |x|^2\big|w_n(\check{t})\big|^2\,\mathrm{d}x + C_0\, e^{-\frac{\lambda(v_* t)^2}{4}}\left(\int |x|^2\big|w_n(\check{t})\big|^2\,\mathrm{d}x\right)^{\frac12}. \tag{3.4.3}$$

Plugging (3.4.2) and (3.4.3) into (3.4.1), and dividing the whole inequality by $\big\||x|\big|w_n(\check{t})\big|\big\|_{L^2}$, we obtain

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\big\||x|\big|w_n(\check{t})\big|\big\|_{L^2}\right| \le C_0\, e^{-\frac{\lambda(v_* t)^2}{4}} + 2\lambda\big\||x|\big|w_n(\check{t})\big|\big\|_{L^2}.$$

Hence, in the same way as in the proof of Proposition 3.2.1, the Gronwall lemma backward in time between T_n and $\check{t}$ and the fact that $w_n(T_n) = 0$ yields for all $t \in [0, T_n - T]$ (and still with $\check{t} = T + t$):

$$\big\|w_n(\check{t})\big\|_{\mathcal{F}(H^1)} \le C_0\, e^{-\frac{\lambda(v_* t)^2}{4}}. \qquad \square$$

3.5. Compactness for the multi-gaussian

This section is devoted to the proof of the Compactness property 3.2.2 for the multi-gaussian case. Since we do not have any bound for the $\mathcal{F}(H^1)$ norm, the proof is here completely similar to that in [116, 53] for example. However, not only in order to be able to perform the same proof but also in order to have a limit in W, we need $u_n(T)$ to be uniformly bounded in W.

Lemma 3.5.1. *$E(u_n)$ is uniformly bounded in n. In particular, u_n is uniformly bounded in $\mathcal{C}_b(\mathbb{R}, W)$.*

Proof. Since we know that the energy is independent in time, we only have to prove for the first part that $E(B(T_n))$ is bounded. First of all, the H^1 norm of $B(T_n)$ is obviously bounded since:

$$\|B(T_n)\|_{H^1} \leq \sum_k \|B_k(T_n)\|_{H^1} \leq \sum_k \left(\|B_k(T_n)\|_{L^2} + \|\nabla B_k(T_n)\|_{L^2}\right).$$

Moreover, $B(T_n)$ is also bounded in L^1:

$$\begin{aligned}\|B(T_n)\|_{L^1} \leq \sum_k \|B_k(T_n)\|_{L^1} &\leq C_0 \sum_k \left\| \exp\left[-\frac{1}{2}(x - x_k - v_k T_n)^\top \operatorname{Re} A(t)(x - x_k - v_k T_n)\right]\right\|_{L^1} \\ &\leq C_0 \left\| e^{-\frac{\sigma_- |x|^2}{2}} \right\|_{L^1} \leq C_0.\end{aligned}$$

Therefore, we claim that $\int |B(T_n)|^2 \ln|B(T_n)|^2$ is uniformly bounded and so is $E(B(T_n))$. Indeed, there holds

$$\int |B(T_n)|^2 \left|\ln|B(T_n)|^2\right| \leq C_0 \left(\int |B(T_n)| + \int |B(T_n)|^{2+\frac{1}{d}}\right) \leq C_0 \left(\|B(T_n)\|_{L^1} + \|B(T_n)\|_{H^1}^{2+\frac{1}{d}}\right),$$

thanks to Sobolev embedding. Therefore, $E(u_n) = E(B(T_n))$ is uniformly bounded. Thus, we can derive that $\nabla u_n(t)$ is uniformly bounded both in n and t from this and the fact that

$$E_n^+(t) \coloneqq \frac{1}{2}\|\nabla u_n(t)\|_{L^2}^2 + \lambda \|u_n(t)\|_{L^2}^2 + \lambda \int_{|u_n(t)| \leq 1} |u_n(t)|^2 \left|\ln |u_n(t)|^2\right| \mathrm{d}x$$

satisfies

$$0 \leq \frac{1}{2}\|\nabla u_n(t)\|_{L^2}^2 \leq E_n^+(t) = E(u_n) + \lambda \int_{|u_n(t)|>1} |u_n(t)|^2 \left|\ln |u_n(t)|^2\right| \mathrm{d}x. \tag{3.5.1}$$

Indeed, by Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}L_n^+(t) = \int_{|u_n(t)|>1} |u_n(t)|^2 \left|\ln |u_n(t)|^2\right| \mathrm{d}x &\leq C_0 \int |u_n(t)|^{2+\frac{1}{2d}} \mathrm{d}x \\ &\leq C_0 \|u_n(t)\|_{L^2}^{1+\frac{1}{2d}} \|\nabla u_n(t)\|_{L^2} \\ &\leq C_0 \|\nabla u_n(t)\|_{L^2},\end{aligned}$$

since the L^2 norm of u_n is uniformly bounded. Thus, putting this into (3.5.1) leads to:

$$\frac{1}{2}\|\nabla u_n(t)\|_{L^2}^2 \leq C_0(1 + \|\nabla u_n(t)\|_{L^2}).$$

Hence, $\|\nabla u_n(t)\|_{L^2}$ is bounded uniformly in n and t, and so is $L_n^+(t)$, therefore so is $E_n^+(t)$. This yields that

$$\int |u_n(t)|^2 \left|\ln |u_n(t)|^2\right| \mathrm{d}x \text{ is bounded uniformly in } t \text{ and } n.$$

Thus, $u_n(t)$ is bounded in $W(\mathbb{R}^d)$ uniformly in t and n. □

Proof of Proposition 3.2.2. We already have a uniform boundedness of $u_n(T)$ in H^1. In order to get compactness in L^2, we shall prove that $u_n(T)$ is compact at infinity. Choose $\delta > 0$. We want to show that there exists r_δ such that for any n we have

$$\int_{|x|>r_\delta} |u_n(T,x)|^2 \mathrm{d}x < \delta.$$

Let $T_\delta \geq 1$ such that

$$e^{-\frac{\sigma_-(v_* T_\delta)^2}{4}} \leq \sqrt{\frac{\delta}{8}},$$

so that, thanks to Proposition 3.2.1, there holds for any $n \in \mathbb{N}$,

$$\|u_n(T+T_\delta,.) - B(T+T_\delta,.)\|_{L^2}^2 \leq \frac{\delta}{8}.$$

The members of B are Gaussians, so we can find $\bar{r}_\delta$ such that

$$\int_{|x|>\bar{r}_\delta} |B(T+T_\delta,x)|^2 \, \mathrm{d}x < \frac{\delta}{8}.$$

Therefore we can infer from the two previous estimates that for all $n \in \mathbb{N}$

$$\int_{|x|>\bar{r}_\delta} |u_n(T+T_\delta,x)|^2 \, \mathrm{d}x < \frac{\delta}{2}.$$

To transfer this property up to T, we use a cut-off function. Take $\hat{r}_\delta > 0$ to be fixed later and a $\mathcal{C}^1$ cut-off function $\chi : \mathbb{R} \to \mathbb{R}$ such that

$$\chi(s) = 0 \text{ for } s < 0, \qquad \chi(s) = 1 \text{ for } s > 1, \qquad \chi(s) \in [0,1] \text{ for } s \in \mathbb{R}.$$

Now set

$$V(t) := \int |u_n(t)|^2 \chi\left(\frac{|x| - \bar{r}_\delta}{\hat{r}_\delta}\right) \mathrm{d}x.$$

Thanks to the previous estimate, we already know that $V(T+T_\delta) < \frac{\delta}{2}$. Moreover, since u_n satisfies (3.1.1), it is easy to compute

$$V'(t) = \frac{2}{\hat{r}_\delta} \int \operatorname{Im}\left(\overline{u_n}(t,x)\frac{x}{|x|} \cdot \nabla u_n(t,x)\right) \chi'\left(\frac{|x| - \bar{r}_\delta}{\hat{r}_\delta}\right) \mathrm{d}x.$$

Thanks to Lemma 3.5.1, we know that $u_n(t)$ is uniformly bounded in H^1. Hence, the previous integral is uniformly bounded and thus we have

$$|V'(t)| \leq \frac{C_0}{\hat{r}_\delta}.$$

We choose now $\hat{r}_\delta$ such that $\frac{C_0}{\hat{r}_\delta} T_\delta < \frac{\delta}{2}$. Hence,

$$|V(T) - V(T+T_\delta)| < \frac{\delta}{2},$$

and therefore

$$V(T) < \delta.$$

We infer from the definition of χ and with $r_\delta = \bar{r}_\delta + \hat{r}_\delta$ that for all $n \in \mathbb{N}$,

$$\int_{|x|>r_\delta} |u_n(T_{\text{in}},x)|^2 \, \mathrm{d}x < \delta,$$

which is the desired conclusion.

Therefore, we get compactness in L^2: there exists a $u_{\text{in}} \in H^1$ such that $u_n(T) \to u_{\text{in}}$ (up to a subsequence) in L^2 as $n \to \infty$. Moreover, $u_n(T)$ is uniformly bounded in W which is a reflexive Banach space when endowed with a Luxembourg type norm (see [38]), so $u_{\text{in}} \in W$. □

3.6. Rigidity property

In this section, we prove the claims made in Remarks 3.1.12 and 3.1.15, which can be summarized as follows:

Lemma 3.6.1. *For any solution v to* (3.1.1)*, either $v = u$ the multi-gaussian constructed above, either there exists $T_1 > 0$ and $C_1 > 0$ such that there holds for all $t \geq T_1$*

$$\left\| v(t) - \sum B_k(t) \right\|_{L^2} \geq C_1 e^{-2\lambda t}.$$

This lemma uses Lemma 3.1.3 backwards, which is also a rigidity property for any solution to (3.1.1).

Proof. Take v solution to (3.1.1). Applying Lemma 3.1.3 with $s = 0$, we have

$$\|v(t) - u(t)\|_{L^2} \geq \|v(0) - u(0)\|_{L^2} e^{-2\lambda t}.$$

Thanks to (3.1.8), we get for all $t \geq 0$:

$$\begin{aligned}\left\|v(T+t) - \sum B_k(T+t)\right\|_{L^2} &\geq \|v(T+t) - u(T+t)\|_{L^2} - \left\|u(T+t) - \sum B_k(T+t)\right\|_{L^2} \\ &\geq \|v(0) - u(0)\|_{L^2} e^{-2\lambda T} e^{-2\lambda t} - e^{-\frac{\sigma_-(v_* t)^2}{4}},\end{aligned}$$

and the conclusion easily follows. □

Appendix

3.A. Proof of Lemma 3.3.9

Before proving this lemma, we prove Lemma 3.2.9:

Proof of Lemma 3.2.9. With a radial change of variables, we get

$$M_{d,n} = C_d \int_R^\infty r^{n+d-1} e^{-\gamma r^2}\,\mathrm{d}r.$$

For $j \in \mathbb{N}$, set

$$I_j := \int_R^\infty r^j e^{-\gamma r^2}\,\mathrm{d}r.$$

For $j = 0$, we have:

$$I_0 < \int_R^\infty \frac{r}{R} e^{-\gamma r^2}\,\mathrm{d}r = \frac{1}{R}\left[-\frac{e^{-\gamma r^2}}{2\gamma}\right]_R^\infty = \frac{1}{2\gamma R} e^{-\gamma R^2}.$$

For $j = 1$, there holds

$$I_1 = \frac{1}{2\gamma} e^{-\gamma R^2}.$$

For the case $j \geq 2$, we get:

$$\begin{aligned} I_j &\leq \int_R^\infty r^{j-1} \cdot r\, e^{-\gamma |r|^2}\,\mathrm{d}r \\ &\leq \left[-\frac{r^{j-1}}{2\gamma} e^{-\gamma |r|^2}\right]_R^\infty + (j-1)\int_R^\infty \frac{r^{j-2}}{2\gamma} e^{-\gamma |r|^2}\,\mathrm{d}r \\ &\leq \frac{R^{j-1}}{2\gamma} e^{-\gamma R^2} + \frac{j-1}{2\gamma} I_{j-2} \\ &\leq \frac{R^{j-1}}{2\gamma} e^{-\gamma R^2} + \frac{j-1}{2} R^2\, I_{j-2}. \end{aligned}$$

From this, we can prove by induction that

$$I_j \leq C_j \frac{R^{j-1}}{\gamma} e^{-\gamma R^2}.$$

The conclusion readily follows. □

Proof of Lemma 3.3.9. We recall that $\psi_j(t', x) \equiv 1$ for $x \in \mathcal{B}_j(t') := \mathcal{B}(x_j^*(t'), \frac{v_* t}{2} + 1)$, $0 \leq \psi_j(t') \leq 1$ and $\|\partial_x \psi_j(t')\|_{L^\infty} \leq 1$ so that the quantities of the first two estimates are all bounded by the H^1 norm on $\mathbb{R} \setminus \mathcal{B}_j(t')$ of $G_j(t')$ up to a multiplicative constant C_0 ($\left\|D^3_{xxx}\psi_k(t', x)\right\|$ is uniformly bounded in $x \in \mathbb{R}^d$ and $t \geq 0$). Then, setting $\mathcal{B}_0(t') := \mathcal{B}(0, \frac{v_* t}{2} + 1)$, we easily compute

$$\begin{aligned} \int_{\mathbb{R}^d \setminus \mathcal{B}_j(t')} (|G_j(t')|^2 + |\nabla G_j(t')|^2)\,\mathrm{d}x &= C_0 \int_{\mathbb{R}^d \setminus \mathcal{B}_j(t')} \left(1 + \left|i v_j - 2\lambda(x - x_j^*(t'))\right|^2\right) \exp\left[-2\lambda \left|x - x_j^*(t')\right|^2\right] \mathrm{d}x \\ &\leq C_0 \int_{\mathcal{B}_0(t')^\complement} \left(1 + |y|^2\right) \exp\left[-2\lambda |y|^2\right] \mathrm{d}y. \end{aligned}$$

Using Lemma 3.2.9, as soon as $\xi(t) = \frac{v_* t}{2} + 1 \geq (2\lambda)^{-\frac{1}{2}}$, we get

$$\int_{\mathbb{R}^d \setminus \mathcal{B}_j(t')} (|G_j(t')|^2 + |\nabla G_j(t')|^2)\,\mathrm{d}x \leq C_0 \Big(\xi(t)^{d-2} + \xi(t)^d\Big) \exp\Big[-2\lambda\,\xi(t)^2\Big] = o\Big(e^{-\nu t} e^{-\frac{\lambda (v_* t)^2}{2}}\Big),$$

which leads to the first two estimates of the lemma. The third estimate can also be deduced from a similar computation.

As for the fourth estimate, there also holds in the same way:

$$\begin{aligned}
&\int_{\mathbb{R}^d \setminus \mathcal{B}_j(t')} (\left|x - x_j^*(t')\right|^4 + \left|x - x_j^*(t')\right|^6) \left|G_j(t')\right|^2 \mathrm{d}x \\
&\qquad \leq C_0 \int_{\mathbb{R}^d \setminus \mathcal{B}_j(t')} (\left|x - x_j^*(t')\right|^4 + \left|x - x_j^*(t')\right|^6) \exp\Big[-2\lambda \left|x - x_j^*(t')\right|^2\Big]\,\mathrm{d}x \\
&\qquad \leq C_0 \int_{\mathcal{B}_0(t')^{\complement}} (|y|^4 + |y|^6) \exp\Big[-2\lambda\,|y|^2\Big]\,\mathrm{d}y \\
&\qquad \leq C_0\,(\xi(t)^{d+2} + \xi(t)^{d+4}) \exp\Big[-2\lambda\,\xi(t)^2\Big] = o\Big(e^{-\nu t} e^{-\frac{\lambda (v_* t)^2}{2}}\Big),
\end{aligned}$$

by using again Lemma 3.2.9. □

3.B. Proof of Proposition 3.4.2

To prove this Proposition, we use a result of [69] giving a pointwise estimate for $\left|G \ln|G|^2 - \sum_k G_k \ln|G_k|^2\right|$. We recall it here in a simplified way which fits our case:

Lemma 3.B.1 ([69, Corollary 3.7])**.** *For $N \in \mathbb{N}^*$, $\lambda > 0$, $x_k \in \mathbb{R}^d$, $\omega_k \in \mathbb{R}$ and $\theta_k : \mathbb{R} \to \mathbb{R}$ a real measurable function for $k = 1, \dots, N$, and g_k such that for all $x \in \mathbb{R}^d$*

$$g_k(x) = \exp\Big[i\theta_k(x) + \omega_k - \lambda |x - x_j|^2\Big],$$

set

$$g(x) = \sum_{k=1}^{N} g_k(x).$$

If

$$\varepsilon := \Big(\min_{k \neq j} |x_j - x_k|\Big)^{-1} < \varepsilon_0,$$

then for any $j \in \{1, \dots, N\}$ and for all $x \in \mathbb{R}$

$$\left| g(x) \ln|g(x)|^2 - \sum_{k=1}^{N} g_k(x) \ln|g_k(x)|^2 \right| \leq 2 \sum_{k \neq j} |g_k(x)| \Big[\delta\omega_j + \delta\omega_k + 3 + 2\ln N + \lambda |x - x_k|^2 + \lambda |x - x_j|^2\Big], \tag{3.B.1}$$

where $\delta\omega_j := \max_\ell \omega_\ell - \omega_j$

Proof of Proposition 3.4.2. Our G_k satisfy the assumptions of Lemma 3.B.1, so that (3.B.1) gives here for all $t \geq 0$, $j \in \{0, \dots, N\}$ and $x \in \mathbb{R}$:

$$\begin{aligned}
\left| G(\check{t}, x) \ln\left|G(\check{t}, x)\right|^2 - \sum_k G_k(\check{t}, x) \ln\left|G_k(\check{t}, x)\right|^2 \right| &\leq C_0 \sum_{k \neq j} \left|G_k(\check{t}, x)\right| \Big[1 + \left|x - x_k^*(\check{t})\right|^2 + \left|x - x_j^*(\check{t})\right|^2\Big] \\
&\leq C_0 \sum_{k \neq j} \left|G_k(\check{t}, x)\right| \Big[1 + t^2 + \left|x - x_k^*(\check{t})\right|^2\Big].
\end{aligned}$$

Thus, multiplying by $|x|$ and ψ_j and taking the L^2 norm leads to:

$$\left\| \psi_j(\check{t}) |x| \left| G \ln|G|^2 - \sum_k G_k \ln|G_k|^2 \right| \right\|_{L^2} \leq C_0 \left\| \psi_j(\check{t}) |x| \sum_{k \neq j} \left|G_k(\check{t})\right| \Big[1 + t^2 + \left|x - x_k^*(\check{t})\right|^2\Big] \right\|_{L^2}$$

$$
\begin{aligned}
&\leq C_0 \left\| \psi_j(\check{t}) \Big(\big|x - x_k^*(\check{t})\big| + \big|x_k^*(\check{t})\big| \Big) \sum_{k \neq j} \big|G_k(\check{t})\big| \Big[1 + t^2 + \big|x - x_k^*(\check{t})\big|^2 \Big] \right\|_{L^2} \\
&\leq C_0 \left\| \psi_j(\check{t}) \Big(\big|x - x_k^*(\check{t})\big| + C_0\, t \Big) \sum_{k \neq j} \big|G_k(\check{t})\big| \Big[1 + t^2 + \big|x - x_k^*(\check{t})\big|^2 \Big] \right\|_{L^2} \\
&\leq C_0 (1 + t^3) \sum_{k \neq j} \left(\big\| \psi_j(\check{t}) \big|G_k(\check{t})\big| \big\|_{L^2} + \Big\| \psi_j(\check{t}) \big|x - x_k^*(\check{t})\big|^3 \big|G_k(\check{t})\big| \Big\|_{L^2} \right), \\
&\leq C_0 (1 + t^3) \sum_{k \neq j} \left(\big\| G_k(\check{t}) \big\|_{L^2(\psi_j(\check{t})\, \mathrm{d}x)} + \Big\| \big|x - x_k^*(\check{t})\big|^3 \big|G_k(\check{t})\big| \Big\|_{L^2(\psi_j(\check{t})\, \mathrm{d}x)} \right).
\end{aligned}
$$

Then, using Corollary 3.3.10 and the last estimate of Lemma 3.3.9, we get

$$
\left\| \psi_j(\check{t}) |x| \left| G \ln|G|^2 - \sum_k G_k \ln|G_k|^2 \right| \right\|_{L^2} \leq C_0\, e^{-\frac{\lambda (v_* t)^2}{4}}.
$$

Thus, we get the result by using the fact that $\sum_j \psi_j = 1$:

$$
\begin{aligned}
\left\| |x| \left| G \ln|G|^2 - \sum_k G_k \ln|G_k|^2 \right| \right\|_{L^2} &= \left\| \sum_j \psi_j(\check{t}) |x| \left| G \ln|G|^2 - \sum_k G_k \ln|G_k|^2 \right| \right\|_{L^2} \\
&\leq \sum_j \left\| \psi_j(\check{t}) |x| \left| G \ln|G|^2 - \sum_k G_k \ln|G_k|^2 \right| \right\|_{L^2} \\
&\leq C_0\, e^{-\frac{\lambda (v_* t)^2}{4}}.
\end{aligned}
$$

□

Chapitre 4

WKB analysis of the Logarithmic Nonlinear Schrodinger Equation in an analytic framework

Abstract. We are interested in a WKB analysis of the Logarithmic Non-Linear Schrödinger Equation with "Riemann-like" variables in an analytic framework in semiclassical regime. We show that the Cauchy problem is locally well posed uniformly in the semiclassical constant and that the semiclassical limit can be performed. In particular, our framework is not only compatible with the Gross-Pitaevskii equation with logarithmic nonlinearity, but also allows initial data (and solutions) which can converge to 0 at infinity.

4.1. Introduction

4.1.1. Setting

We are interested in the *Logarithmic Non-Linear Schrödinger Equation* (also called logNLS)

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \lambda \ln |u^\varepsilon|^2 u^\varepsilon, \qquad u^\varepsilon(0) = u^\varepsilon_{\text{in}}, \tag{4.1.1}$$

with $x \in \mathbb{R}^d$, $d \geq 1$, $\lambda \in \mathbb{R} \setminus \{0\}$, $\varepsilon > 0$. This equation was introduced as a model of nonlinear wave mechanics and in nonlinear optics ([23], see also [25, 95, 98, 102, 59, 105, 165]). The case $\lambda > 0$ (whose study of the Cauchy problem goes back to [40, 86]) was studied by R. Carles and I. Gallagher who made explicit an unusually faster dispersion with a universal behaviour of the modulus of the solution (see [33]). The knowledge of this behaviour was recently improved with a convergence rate but also extended through the semiclassical limit in [67]. On the other hand, the case $\lambda < 0$ seems to be the most interesting from a physical point of view and has been studied formally and rigorously (see for instance [38, 58, 98, 36, 69, 68]).

This paper addresses the semiclassical limit of (4.1.1) for general $\lambda \neq 0$ through WKB analysis in an analytic framework. For this, we first address the Cauchy problem of the system given by this WKB analysis (see (4.1.15)) and give a local Cauchy theory independent of $\varepsilon \in [0, 1]$. Then, we prove that the solutions for $\varepsilon > 0$ converge when $\varepsilon \to 0$ to the solution constructed for $\varepsilon = 0$ as expected. Last, we address the complete convergence of the wave function u^ε as $\varepsilon \to 0$.

4.1.2. The WKB analysis for NLS

In the case $\lambda > 0$, R. Carles and A. Nouri [36] have performed a WKB analysis of this equation: for initial data of the form $u^\varepsilon_{\text{in}} = \sqrt{\rho_{\text{in}}}\, e^{i\frac{\phi_{\text{in}}}{\varepsilon}}$ (in general dimension d), one can seek u^ε under the form $u^\varepsilon = a^\varepsilon e^{i\frac{\phi^\varepsilon}{\varepsilon}}$ where $a^\varepsilon \in \mathbb{C}$ and $\phi^\varepsilon \in \mathbb{R}$ satisfy:

$$\begin{cases} \partial_t \phi^\varepsilon + \frac{1}{2}\nabla\phi^\varepsilon \cdot \nabla\phi^\varepsilon + \lambda \ln |a^\varepsilon|^2 = 0, & \phi^\varepsilon(0) = \phi_{\text{in}}, \\ \partial_t a^\varepsilon + \nabla\phi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta\phi^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon, & a^\varepsilon(0) = \sqrt{\rho_{\text{in}}}. \end{cases} \tag{4.1.2}$$

Note that allowing a^ε to be complex-valued (even though $\sqrt{\rho_{\text{in}}}$ is real-valued) gives a degree of freedom to dispatch terms from (4.1.1) into this system. Then, they follow the choice introduced by Grenier which is more robust than the Madelung transform when semiclassical limit is considered (see [31]). From this system, one usually defines

$$v^\varepsilon := \nabla\phi^\varepsilon. \tag{4.1.3}$$

This relation is also equivalent to (see [36])

$$\phi^\varepsilon(t, x) = \phi_{\text{in}}(x) - \int_0^t \Big(\frac{1}{2}|v^\varepsilon(\tau, x)|^2 + \lambda \ln |a^\varepsilon(\tau, x)|^2\Big)\,\mathrm{d}\tau,$$

so that, along with a^ε, determining ϕ^ε turns out to be equivalent to determining v^ε solution to

$$\left\{\begin{array}{ll} \partial_t v^\varepsilon + (v^\varepsilon \cdot \nabla) v^\varepsilon + \lambda \nabla \Big(\ln |a^\varepsilon|^2\Big) = 0, & v^\varepsilon(0) = \nabla \phi_{\text{in}}, \\ \partial_t a^\varepsilon + v^\varepsilon \cdot \nabla a^\varepsilon + \dfrac{1}{2} a^\varepsilon \operatorname{div} v^\varepsilon = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon, & a^\varepsilon(0) = \sqrt{\rho_{\text{in}}}. \end{array}\right. \tag{4.1.4}$$

Remark 4.1.1. To get this system, we have used the fact that

$$\frac{1}{2}\nabla(|v^\varepsilon|^2) = (v^\varepsilon \cdot \nabla) v^\varepsilon, \tag{4.1.5}$$

due to the fact that v^ε is a gradient. All across this paper, we will use the general fact that, for an irrotational field f, one has

$$\frac{1}{2}\nabla(|f|^2) = (f \cdot \nabla) f.$$

The semiclassical limit $\varepsilon \to 0$ for u^ε relates classical and quantum wave equations and is expected to be described by the laws of hydrodynamics (see e.g. [73, 79, 72, 60]). In particular, passing formally to the limit $\varepsilon \to 0$ in (4.1.4) leads to:

$$\left\{\begin{array}{ll} \partial_t v + (v \cdot \nabla) v + \lambda \nabla \Big(\ln |a|^2\Big) = 0, & v(0) = \nabla \phi_{\text{in}}, \\ \partial_t a + v \cdot \nabla a + \dfrac{1}{2} a \operatorname{div} v = 0, & a(0) = \sqrt{\rho_{\text{in}}}. \end{array}\right. \tag{4.1.6}$$

which is the symmetrized version of the isothermal Euler system ($\varepsilon = 0$) with $\rho = |a|^2$ (see [45, 113]):

$$\left\{\begin{array}{l} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \lambda \nabla \rho = 0. \end{array}\right. \tag{4.1.7}$$

The WKB analysis is not exclusive to (4.1.1), it has been used a lot for general non-linear Schrödinger equations. For instance, it has been discussed in [26] for general nonlinearity of the form

$$i\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \varepsilon^\kappa f(|u^\varepsilon|^2) u^\varepsilon. \tag{4.1.8}$$

In particular, the WKB type analysis is justified for $\kappa \geq 1$, which corresponds to a weak nonlinearity. When $\kappa = 0$, the mathematical analysis of the semiclassical limit for nonlinear Schrodinger equations has been well developed for two cases: for analytic initial data (see for instance [73, 140, 141]) and for initial data in some Sobolev space with a defocusing nonlinearity so that the analogue of (4.1.4) is hyperbolic symmetric, possibly with a change of variables (see [79, 2, 49]). It was also extended to the case of generalized derivative nonlinear Schrodinger equations (in dimension $d = 1$)

$$i\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon + i\frac{\varepsilon}{2}\partial_x (g(|u^\varepsilon|^2) u^\varepsilon) = \varepsilon^\kappa f(|u^\varepsilon|^2) u^\varepsilon.$$

In [61], the semiclassical analysis for this equation relies on the assumption

$$\partial_x \phi^\varepsilon \, g' > 0, \qquad f \equiv 0,$$

and was generalized by [60] to the case

$$\partial_x \phi^\varepsilon \, g' + f' > 0.$$

These assumptions are made to ensure hyperbolicity, but have the strong drawback to involve the solution itself. However, hyperbolicity is not needed when one works with analytic functions ([34]). In this context, the semiclassical limit for (4.1.8) (with $\kappa = 0$) was studied by [73, 141], thanks to some tools developed by J. Sjöstrand [135].

On the other hand, WKB analysis is also useful for the study of the Gross-Pitaevskii equation, for instance in the context where initial data do not necessarily decay to zero at infinity. The Cauchy problem ([71, 74]) and the semiclassical limit ([108]) of (4.1.8) with $f(y) = y - 1$ and $\kappa = 0$ for instance have already been studied. In this case, the Hamiltonian structure yields that

$$\mathcal{E}(u^\varepsilon) = \varepsilon^2 \|\nabla u^\varepsilon(t)\|_{L^2}^2 + \Big\| |u^\varepsilon(t)|^2 - 1 \Big\|_{L^2}^2$$

is independent of time, at least formally, which leads to a natural energy space,

$$E = \Big\{ u \in H^1_{\text{loc}}; \nabla u \in L^2, |u|^2 - 1 \in L^2 \Big\},$$

to study the Cauchy problem (see also [21]). The modulus of functions in this space morally goes to 1 at infinity. For more general initial data which are bounded but may not be in this space (for instance if they have several limits at infinity), P. E. Zhidkov introduced in the one-dimensional case in [163, 164] the so-called Zhidkov spaces:

$$X^s = \left\{u \in L^\infty ; \nabla u \in H^{s-1}\right\}, \qquad s > \frac{d}{2}.$$

The study of theses spaces was generalized in the multidimensional case by C. Gallo ([71]). They were also used by T. Alazard and R. Carles [3] in their WKB analysis for the Gross-Pitaevskii equation. R. Carles and A. Nouri [36] have also shown that, for initial data in Zhidkov spaces bounded away from vacuum, the Wigner measure of u^ε solution to (4.1.1) weakly converges to a monokinetic measure $f = \rho(t,x) \otimes \delta_{\xi = v(t,x)}$ such that (ρ, v) satisfies the isothermal Euler system, thanks to the WKB analysis described before.

4.1.3. Riemann invariants

The isothermal Euler system (4.1.7) has been studied a lot in different contexts (for example [143, 22, 62, 47, 11]). B. Riemann solved the "Riemann problem" for this equation in his memoir to the Royal Academy of Sciences of Göttinger (1860) (see [133]). In dimension $d = 1$ and in the case $\lambda > 0$, he introduced the so-called Riemann invariants $w_1 = v + \sqrt{2\lambda}\ln\rho$ and $w_2 = v - \sqrt{2\lambda}\ln\rho$. Then, he proved that the necessary and sufficient condition for the solution to exist for all positive times is that w_1 (resp. w_2) is non-decreasing (resp. non-increasing).

This shows that the good unknown to be considered would rather be $\ln\rho$ in (4.1.7), or $\ln a$ in (4.1.6). In particular, we should therefore consider $\ln a^\varepsilon$ in (4.1.4). This intuition is strengthened by the fact that dividing the second equation by a^ε in system (4.1.4) gives

$$\frac{\partial_t a^\varepsilon}{a^\varepsilon} + v^\varepsilon \cdot \frac{\nabla a^\varepsilon}{a^\varepsilon} + \frac{1}{2}\operatorname{div} v^\varepsilon = i\frac{\varepsilon}{2}\frac{\Delta a^\varepsilon}{a^\varepsilon},$$

which could formally be written in terms of $\ln a^\varepsilon$ only, since we also have (at least for f real)

$$\frac{\Delta f}{f} = \Delta(\ln f) + \nabla(\ln f)\cdot\nabla(\ln f).$$

Moreover, in the first equation in (4.1.4), if formally $f = \ln a^\varepsilon$ *i.e.* $a^\varepsilon = e^f$, then

$$\ln|a^\varepsilon|^2 = 2\operatorname{Re} f.$$

However, in the latter, contrary to a, a^ε is complex, a fact which may lead to some problems when defining $\ln a^\varepsilon$. Still, we can override this difficulty. Indeed, a^ε is defined this way only to get u^ε solution to (4.1.1). Instead of defining a^ε, one can try to directly define ψ^ε (along with ϕ^ε) such that $u^\varepsilon = e^{\frac{\psi^\varepsilon}{2} + i\frac{\phi^\varepsilon}{\varepsilon}}$ is solution to (4.1.1). For this, we first assume $\rho^\varepsilon_{\text{in}} = e^{\psi^\varepsilon_{\text{in}}}$, so that $u^\varepsilon_{\text{in}} = e^{\frac{\psi^\varepsilon_{\text{in}}}{2} + i\frac{\phi^\varepsilon_{\text{in}}}{\varepsilon}}$ (note that we allow the initial data to depend on ε with suitable conditions which will be made explicit later). Thus we can seek u^ε under the form $u^\varepsilon = e^{\frac{\psi^\varepsilon}{2} + i\frac{\phi^\varepsilon}{\varepsilon}}$ with $\psi^\varepsilon \in \mathbb{C}$ and $\phi^\varepsilon \in \mathbb{R}$ such that:

$$\begin{cases} \partial_t\phi^\varepsilon + \frac{1}{2}\nabla\phi^\varepsilon\cdot\nabla\phi^\varepsilon + \lambda\operatorname{Re}\psi^\varepsilon = 0, & \phi^\varepsilon(0) = \phi^\varepsilon_{\text{in}}, \\ \partial_t\psi^\varepsilon + \nabla\phi^\varepsilon\cdot\nabla\psi^\varepsilon + \Delta\phi^\varepsilon = i\frac{\varepsilon}{2}\Big(\Delta\psi^\varepsilon + 2\,\nabla\psi^\varepsilon\cdot\nabla\psi^\varepsilon\Big), & \psi^\varepsilon(0) = \psi^\varepsilon_{\text{in}}. \end{cases} \tag{4.1.9}$$

Note that the pseudo scalar product in the second equation is defined for $a, b \in \mathbb{C}^d$ by

$$a \cdot b = \sum_i a_i\, b_i,$$

and therefore $a \cdot a$ is not necessarily real. In the same way as for passing from (4.1.2) to (4.1.4), we can define $v^\varepsilon := \nabla\phi^\varepsilon$ (4.1.3), which leads to

$$\begin{cases} \partial_t v^\varepsilon + (v^\varepsilon\cdot\nabla)v^\varepsilon + \lambda\nabla\Big(\operatorname{Re}\psi^\varepsilon\Big) = 0, & v^\varepsilon(0) = \nabla\phi^\varepsilon_{\text{in}}, \\ \partial_t\psi^\varepsilon + v^\varepsilon\cdot\nabla\psi^\varepsilon + \operatorname{div} v^\varepsilon = i\frac{\varepsilon}{2}\Big(\Delta\psi^\varepsilon + 2\,\nabla\psi^\varepsilon\cdot\nabla\psi^\varepsilon\Big), & \psi^\varepsilon(0) = \psi^\varepsilon_{\text{in}}. \end{cases} \tag{4.1.10}$$

Moreover, (4.1.3) is here equivalent to

$$\phi^\varepsilon(t,x) = \phi^\varepsilon_{\text{in}}(x) - \int_0^t \Big(\frac{1}{2}|v^\varepsilon(\tau,x)|^2 + \lambda\operatorname{Re}\psi^\varepsilon(\tau,x)\Big)\,\mathrm{d}\tau. \tag{4.1.11}$$

As soon as the initial data converges, passing formally to the semiclassical limit $\varepsilon \to 0$ in system (4.1.10) yields:

$$\begin{cases} \partial_t v + (v\cdot\nabla)v + \lambda\nabla\psi = 0, & v(0) = \nabla\phi_{\text{in}}, \\ \partial_t\psi + v\cdot\nabla\psi + \operatorname{div} v = 0, & \psi(0) = \psi_{\text{in}}. \end{cases} \tag{4.1.12}$$

This system is linked to the isothermal Euler system (4.1.7) (and thus also to (4.1.6)) in the sense that it is the same but written in "Riemann-like" variables (with $\rho = e^\psi$).

Remark 4.1.2. If we neglect the convective terms $(v\cdot\nabla)v$ and $v\cdot\nabla\psi$, which are still not regularizing terms and may already lead to shocks (like for the Burgers equation), then we would get the system

$$\begin{cases} \partial_t v + \lambda\,\nabla\psi = 0, \\ \partial_t\psi + \operatorname{div} v = 0. \end{cases} \tag{4.1.13}$$

For instance, ψ would satisfy

$$\partial_{tt}^2\psi - \lambda\Delta\psi = 0,$$

and a similar equation would hold for v. In particular, when $\lambda > 0$, we get the wave equation, which is well posed in L^2-based spaces (*e.g.* $(\psi(0), \partial_t\psi(0)) \in H^1 \times L^2$), or based on Zhidkov spaces for instance. However, if $\lambda < 0$, this equation becomes way more singular. Indeed, for the Fourier transform, defined for every $f \in L^1$ and for every $\xi \in \mathbb{R}^d$ by

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) := \int f(x)e^{-ix\cdot\xi}\,\mathrm{d}x,$$

and then extended for any $f \in L^2$, we get

$$\partial_{tt}^2\hat{\psi} - |\lambda||\xi|^2\hat{\psi} = 0,$$

whose solutions are

$$\hat{\psi}(\xi) = \hat{\psi}(0)\,\cosh(\sqrt{|\lambda|}|\xi|t) + \partial_t\hat{\psi}(0)\,\sinh(\sqrt{|\lambda|}|\xi|t).$$

Thus the Fourier transform is in L^2 for some interval $[0,T]$ only if the initial data are analytic. Hence, one may probably not hope for a Cauchy theory of (4.1.12) (and thus for (4.1.10)) in lower regularities for this case (see for instance [120] in 1D and [107] in higher dimension). Our construction will still work for $\lambda > 0$, so we take the general case $\lambda \neq 0$.

4.1.4. Transformation of the system

Obviously, we have

$$\nabla\Big(\operatorname{Re}\psi^\varepsilon\Big) = \operatorname{Re}\Big(\nabla\psi^\varepsilon\Big)$$

Therefore, the system (4.1.10) does not involve ψ^ε directly, but only derivatives of this function. Therefore, in the same way as for ϕ^ε and v^ε, one can transform the equation for ψ^ε into an equation for

$$\zeta^\varepsilon := \nabla\psi^\varepsilon, \tag{4.1.14}$$

so that (4.1.10) becomes

$$\begin{cases} \partial_t v^\varepsilon + (v^\varepsilon\cdot\nabla)v^\varepsilon + \lambda\,\operatorname{Re}\zeta^\varepsilon = 0, & v^\varepsilon(0) = \nabla\phi^\varepsilon_{\text{in}}, \\ \partial_t\zeta^\varepsilon + \nabla\Big(v^\varepsilon\cdot\zeta^\varepsilon\Big) + \nabla\operatorname{div} v^\varepsilon = i\dfrac{\varepsilon}{2}\Big(\nabla\operatorname{div}\zeta^\varepsilon + 2\,\nabla(\zeta^\varepsilon\cdot\zeta^\varepsilon)\Big), & \zeta^\varepsilon(0) = \nabla\psi^\varepsilon_{\text{in}}. \end{cases} \tag{4.1.15}$$

In the same way as the relation between (4.1.3) and (4.1.11), (4.1.14) is equivalent to

$$\psi^\varepsilon(t,x) = \psi^\varepsilon_{\text{in}}(x) - \int_0^t \Big[v^\varepsilon(\tau,x)\cdot\zeta^\varepsilon(\tau,x) + \operatorname{div} v^\varepsilon(\tau,x) - i\frac{\varepsilon}{2}\Big(\operatorname{div}\zeta^\varepsilon(\tau,x) + 2\,\zeta^\varepsilon(\tau,x)\cdot\zeta^\varepsilon(\tau,x)\Big)\Big]\,\mathrm{d}\tau. \tag{4.1.16}$$

Indeed, it is obvious that, with the definition (4.1.16), $\psi^\varepsilon(0) = \psi^\varepsilon_{\text{in}}$, *i.e.* $\nabla\psi^\varepsilon(0) = \nabla\psi^\varepsilon_{\text{in}} = \zeta^\varepsilon(0)$, and one can easily compute that

$$\partial_t(\nabla\psi^\varepsilon) - \partial_t\zeta^\varepsilon = \nabla\partial_t\psi^\varepsilon - \partial_t\zeta^\varepsilon = 0.$$

4.1.5. Main results

Notations for analytic spaces. All across the paper, we denote

$$\langle\xi\rangle := \sqrt{1+|\xi|^2}.$$

Then, for $\delta, \ell \geq 0$ and $n \in \mathbb{N}^*$, define the analytic spaces (like in [77]):

$$\mathcal{H}^\ell_\delta(\mathbb{R}^d,\mathbb{R}^n) := \Big\{f \in L^2(\mathbb{R}^d,\mathbb{R}^n), \|f\|_{\mathcal{H}^\ell_\delta} < \infty\Big\},$$

where $L^2(\mathbb{R}^d,\mathbb{R}^n)$ designates the functions in L^2 from $\mathbb{R}^d$ with values in $\mathbb{R}^n$ and

$$\|f\|^2_{\mathcal{H}^\ell_\delta} = \int_{\mathbb{R}^d}\langle\xi\rangle^{2\ell}e^{2\delta\langle\xi\rangle}\Big|\hat{f}(\xi)\Big|^2\,\mathrm{d}\xi =: \|f\|^2_{\ell,\delta},$$

and $\hat{f} = \mathcal{F}(f)$ designates the Fourier transform in space variables. We also define the scalar product associated to this norm:

$$\langle f, g\rangle_{\ell,\delta} := \int_{\mathbb{R}^d} \langle\xi\rangle^{2\ell} e^{2\delta\langle\xi\rangle} \hat{f}(\xi) \cdot \overline{\hat{g}(\xi)} \,\mathrm{d}\xi.$$

For simplicity of notations, we will drop $(\mathbb{R}^d, \mathbb{R}^n)$ in the definition of $\mathcal{H}^\ell_\delta$ and also in L^2. When we consider "continuous" $\mathcal{H}^\ell_\delta$ valued functions, these are functions that belong to

$$\mathcal{C}(I, \mathcal{H}^\ell_\delta) := \left\{ f \in \mathcal{C}(I, L^2), \mathcal{F}^{-1}(w_\delta \hat{f}) \in \mathcal{C}(I, \mathcal{H}^\ell_0) = \mathcal{C}(I, H^\ell) \right\},$$

for some interval I and where

$$w_\delta := \exp\Big(\delta\langle\xi\rangle\Big),$$

with $\delta = \delta(t)$ continuous (and even $\mathcal{C}^1$). When $I = [0, T]$, we denote

$$\begin{aligned} \mathcal{C}_T\mathcal{H}^\ell_\delta &:= \mathcal{C}([0,T], \mathcal{H}^\ell_\delta), \\ L^\infty_T\mathcal{H}^\ell_\delta &:= L^\infty((0,T), \mathcal{H}^\ell_\delta), \\ L^2_T\mathcal{H}^\ell_\delta &:= L^2((0,T), \mathcal{H}^\ell_\delta), \\ H^1_T\mathcal{H}^\ell_\delta &:= H^1((0,T), \mathcal{H}^\ell_\delta). \end{aligned}$$

Moreover, we also denote the following norms for any $t \in [0, T]$ as

$$\begin{aligned} |||f|||_{\infty,t,\ell,\delta} &:= \|f\|_{L^\infty_t\mathcal{H}^\ell_\delta} = \sup_{\tau\in(0,t)} \|f(\tau)\|_{\ell,\delta(\tau)}, \\ |||f|||_{2,t,\ell,\delta} &:= \|f\|_{L^2_t\mathcal{H}^\ell_\delta} = \Big(\int_0^t \|f(\tau)\|^2_{\ell,\delta(\tau)} \,\mathrm{d}\tau\Big)^{\frac{1}{2}}. \end{aligned}$$

Finally, for any $M, \ell, \delta > 0$ and $f \in L^\infty_T\mathcal{H}^\ell_\delta \cap L^2_T\mathcal{H}^{\ell+\frac{1}{2}}_\delta \cap \mathcal{C}_T\mathcal{H}^{\ell-\frac{1}{2}}_\delta$, we also define for all $t \in [0, T]$:

$$\mathcal{E}_{M,\ell,\delta}(f)(t) := \|f(t)\|^2_{\ell,\delta(t)} + 2M\, |||f|||^2_{2,t,\ell,\delta}.$$

Main result on $(\zeta^\varepsilon, v^\varepsilon)$. We are interested in system (4.1.15) in an analytic framework. For this, fix $\lambda \neq 0$, $\ell > \frac{d}{2}$ and $\delta_{\text{in}} > 0$ for the rest of this paper. Then, we assume the following:

Assumption 1. *$\psi^\varepsilon_{\text{in}}, \phi^\varepsilon_{\text{in}} \in \mathcal{C}^1(\mathbb{R}^d)$ are such that $\nabla\psi^\varepsilon_{\text{in}} \in \mathcal{H}^\ell_{\delta_{\text{in}}}$ and $\nabla\phi^\varepsilon_{\text{in}} \in \mathcal{H}^{\ell+1}_{\delta_{\text{in}}}$ are uniformly bounded in $\varepsilon \in [0,1]$ in these spaces: there exists ω_{in} such that for all $\varepsilon \in [0,1]$*

$$\|\nabla\psi^\varepsilon_{\text{in}}\|^2_{\ell,\delta_{\text{in}}} + \|\nabla\phi^\varepsilon_{\text{in}}\|^2_{\ell+1,\delta_{\text{in}}} \leq \omega_{\text{in}}.$$

Remark 4.1.3. The initial data $\psi^\varepsilon_{\text{in}}$ and $\phi^\varepsilon_{\text{in}}$ might be unbounded when $|x| \to \infty$, or have different limits at infinity (for instance at $\pm\infty$ in dim $d = 1$, see Section 4.6 and more specifically Lemma 4.6.1). In particular, this means that we allow the initial data $u^\varepsilon_{\text{in}}$ for (4.1.1) or ρ^0_{in} for (4.1.7) to be near vacuum at infinity. This is different from [36], which requires the initial data to be bounded away from zero (along with $\lambda > 0$).

Our first main result is divided into two parts. Under the previous assumptions, the first part addresses the Cauchy problem of (4.1.15). With such a Cauchy theory, we then deal with semiclassical limit by stating that $(\zeta^\varepsilon, v^\varepsilon)$ converge (in some sense) to (ζ^0, v^0) as $\varepsilon \to 0$. For the latter, we define for any $k > 0$ the following constant which depends only on ε:

$$(D^\varepsilon_k)^2 := \left\|\nabla\psi^\varepsilon_{\text{in}} - \nabla\psi^0_{\text{in}}\right\|^2_{k,\delta_{\text{in}}} + \left\|\nabla\phi^\varepsilon_{\text{in}} - \nabla\phi^0_{\text{in}}\right\|^2_{k+1,\delta_{\text{in}}}.$$

Theorem 4.1.4. *For any $(\psi^\varepsilon_{\text{in}}, \phi^\varepsilon_{\text{in}})$ satisfying Assumption 1, there exists $T > 0$, $M > 0$ and $\delta = \delta(t) := \delta_{\text{in}} - Mt$ such that, for all $\varepsilon \in [0,1]$:*

- *There exists a unique solution $(\zeta^\varepsilon, v^\varepsilon) \in L^\infty((0,T), \mathcal{H}^\ell_\delta \times \mathcal{H}^{\ell+1}_\delta) \cap \mathcal{C}([0,T], \mathcal{H}^{\ell-\frac{1}{2}}_\delta \times \mathcal{H}^{\ell+\frac{1}{2}}_\delta) \cap L^2((0,T), \mathcal{H}^{\ell+\frac{1}{2}}_\delta \times \mathcal{H}^{\ell+\frac{3}{2}}_\delta)$ to* (4.1.15).
- *There exists $C > 0$ independent of $\varepsilon \in [0,1]$ such that*

$$|||\zeta^\varepsilon - \zeta^0|||_{\infty,T,\ell-\frac{1}{2},\delta} + |||v^\varepsilon - v^0|||_{\infty,T,\ell+\frac{1}{2},\delta} + |||\zeta^\varepsilon - \zeta^0|||_{2,T,\ell,\delta} + |||v^\varepsilon - v^0|||_{2,T,\ell+1,\delta} \leq C\Big(\sqrt{\varepsilon} + D^\varepsilon_{\ell-\frac{1}{2}}\Big),$$

and, if $\ell > \frac{d+1}{2}$,

$$|||\zeta^\varepsilon - \zeta^0|||_{\infty,T,\ell-1,\delta} + |||v^\varepsilon - v^0|||_{\infty,T,\ell,\delta} + |||\zeta^\varepsilon - \zeta^0|||_{2,T,\ell-\frac{1}{2},\delta} + |||v^\varepsilon - v^0|||_{2,T,\ell+\frac{1}{2},\delta} \leq C\Big(\varepsilon + D^\varepsilon_{\ell-1}\Big).$$

Main result on $(\psi^\varepsilon, \phi^\varepsilon)$. Once we have a Cauchy theory for the system (4.1.15), we can define ψ^ε with (4.1.16). From this definition, it is easy to prove that $\nabla\psi^\varepsilon = \zeta^\varepsilon$ like in Section 4.1.4. In a similar way, we then define ϕ^ε with (4.1.11). However, we can not prove directly that $\nabla\phi^\varepsilon = v^\varepsilon$. Indeed, we need (4.1.5) to hold, *i.e.* that v^ε is irrotational. This is obviously true at $t = 0$. To prove it for $t > 0$, note that taking the curl of the equation on v^ε gives a linear equation on $\operatorname{curl} v^\varepsilon$. Therefore, from the previous result, we also gain a local Cauchy theory for (4.1.9) through the relations (4.1.11) and (4.1.16). Moreover, the semiclassical limit can also be extended to these functions, which leads to define for any $k > 0$ the following constant which depends only on ε:

$$(\tilde{D}_k^\varepsilon)^2 := \left\|\psi_{\text{in}}^\varepsilon - \psi_{\text{in}}^0\right\|_{k,\delta_{\text{in}}}^2 + \left\|\phi_{\text{in}}^\varepsilon - \phi_{\text{in}}^0\right\|_{k+1,\delta_{\text{in}}}^2. \tag{4.1.17}$$

Since the assumptions for the initial data may lead to non-trivial behavior at infinity for $(\psi_{\text{in}}^\varepsilon, \phi_{\text{in}}^\varepsilon)$, we also address the behavior at infinity (in space) of $(\psi^\varepsilon(t), \phi^\varepsilon(t))$ thanks to the relations (4.1.11) and (4.1.16).

Corollary 4.1.5. *For any* $(\psi_{\text{in}}^\varepsilon, \phi_{\text{in}}^\varepsilon)$ *satisfying Assumption 1, there exists* $T > 0$*,* $M > 0$ *and* $\delta = \delta(t) := \delta_{\text{in}} - Mt$ *such that, for all* $\varepsilon \in [0, 1]$*:*

— *There exists a unique solution* $(\psi^\varepsilon, \phi^\varepsilon) \in \mathcal{C}^2([0, T] \times \mathbb{R}^d)^2$ *to* (4.1.9) *such that* $(\nabla\psi^\varepsilon, \nabla\phi^\varepsilon) \in L^\infty((0, T), \mathcal{H}_\delta^\ell \times \mathcal{H}_\delta^{\ell+1}) \cap \mathcal{C}([0, T], \mathcal{H}_\delta^{\ell-\frac{1}{2}} \times \mathcal{H}_\delta^{\ell+\frac{1}{2}}) \cap L^2((0, T), \mathcal{H}_\delta^{\ell+\frac{1}{2}} \times \mathcal{H}_\delta^{\ell+\frac{3}{2}})$.

— *There holds*

$$\psi^\varepsilon - \psi_{\text{in}}^\varepsilon \in H_T^1\mathcal{H}_\delta^{\ell-\frac{1}{2}} \cap \mathcal{C}_T\mathcal{H}_\delta^{\ell+\frac{1}{2}} \cap L_T^\infty\mathcal{H}_\delta^{\ell+1} \cap L_T^2\mathcal{H}_\delta^{\ell+\frac{3}{2}},$$
$$\phi^\varepsilon - \phi_{\text{in}}^\varepsilon - \lambda t\,\psi_{\text{in}}^\varepsilon \in H_T^1\mathcal{H}_\delta^{\ell+\frac{3}{2}} \cap L_T^\infty\mathcal{H}_\delta^{\ell+2} \cap L_T^2\mathcal{H}_\delta^{\ell+\frac{5}{2}}.$$

— *There exists* $C > 0$ *independent of* $\varepsilon \in [0, 1]$ *such that*

$$\left|\!\left|\!\left|\psi^\varepsilon - \psi^0\right|\!\right|\!\right|_{\infty,T,\ell+\frac{1}{2},\delta} + \left|\!\left|\!\left|\phi^\varepsilon - \phi^0\right|\!\right|\!\right|_{\infty,T,\ell+\frac{3}{2},\delta} + \left|\!\left|\!\left|\psi^\varepsilon - \psi^0\right|\!\right|\!\right|_{2,T,\ell+1,\delta} + \left|\!\left|\!\left|\phi^\varepsilon - \phi^0\right|\!\right|\!\right|_{2,T,\ell+2,\delta} \leq C\Big(\sqrt{\varepsilon} + \tilde{D}_{\ell+\frac{1}{2}}^\varepsilon\Big),$$

and, if $\ell > \frac{d+1}{2}$*,*

$$\left|\!\left|\!\left|\psi^\varepsilon - \psi^0\right|\!\right|\!\right|_{\infty,T,\ell,\delta} + \left|\!\left|\!\left|\phi^\varepsilon - \phi^0\right|\!\right|\!\right|_{\infty,T,\ell+1,\delta} + \left|\!\left|\!\left|\psi^\varepsilon - \psi^0\right|\!\right|\!\right|_{2,T,\ell+\frac{1}{2},\delta} + \left|\!\left|\!\left|\phi^\varepsilon - \phi^0\right|\!\right|\!\right|_{2,T,\ell+\frac{3}{2},\delta} \leq C\Big(\varepsilon + \tilde{D}_\ell^\varepsilon\Big).$$

Remark 4.1.6. $a^\varepsilon = e^{\frac{\psi^\varepsilon}{2}}$ satisfies (4.1.2) and (4.1.4) (along with ϕ^ε and v^ε respectively). In particular, for $\varepsilon > 0$, $u^\varepsilon = e^{\frac{\psi^\varepsilon}{2} + i\frac{\phi^\varepsilon}{\varepsilon}}$ is a $\mathcal{C}^2$ solution to (4.1.1) where ϕ^ε is defined by (4.1.11).

Remark 4.1.7. One can add any constant to v^ε, *i.e.* any linear function to ϕ^ε, thanks to the Galilean invariance: for any $c_0 \in \mathbb{R}^d$ and any $(\zeta^\varepsilon, v^\varepsilon)$ solution to (4.1.15), $(\zeta^\varepsilon(t, x - c_0t), v^\varepsilon(t, x - c_0t) + c_0)$ is also solution to (4.1.15), and a similar relation holds for ψ^ε and ϕ^ε. Moreover, the addition of a constant to ψ_{in} gives an explicit behaviour thanks to the effect of scaling for (4.1.1): if u^ε is a solution to (4.1.1) and $\kappa \in \mathbb{R}$, then

$$u^\varepsilon(t, x)\, e^{\kappa - \frac{2it\lambda\kappa}{\varepsilon}}$$

also solves (4.1.1) (with initial datum $e^\kappa u^\varepsilon(0)$). The corresponding relation for $(\psi^\varepsilon, \phi^\varepsilon)$ is that, if $(\psi^\varepsilon, \phi^\varepsilon)$ satisfies (4.1.9), then $(\psi^\varepsilon + 2\kappa, \phi^\varepsilon - 2\lambda\kappa t)$ also satisfies (4.1.9), and this also holds for $\varepsilon = 0$. The second part of Corollary 4.1.5, in particular the term $\lambda t\,\psi_{\text{in}}^\varepsilon$, is therefore consistent with the effect of the scaling.

Remark 4.1.8. In the case where $\psi_{\text{in}}^\varepsilon$ and $\phi_{\text{in}}^\varepsilon$ are independent of ε, the convergences are in $O(\sqrt{\varepsilon})$ and $O(\varepsilon)$ respectively for the semiclassical parts in Theorem 4.1.4 and Corollary 4.1.5. Actually, if $\ell > \frac{d+1}{2}$, the first case can be deduced from the second case (in both the second part of Theorem 4.1.4 and the third one of Corollary 4.1.5) by the fact that $(\zeta^\varepsilon, v^\varepsilon)$ is uniformly bounded (for $\varepsilon \in [0, 1]$) in $L^\infty((0, T), \mathcal{H}_\delta^\ell \times \mathcal{H}_\delta^{\ell+1}) \cap L^2((0, T), \mathcal{H}_\delta^{\ell+\frac{1}{2}} \times \mathcal{H}_\delta^{\ell+\frac{3}{2}})$.

The second part of Corollary 4.1.5 gives useful information about the behavior of $\psi^\varepsilon(t)$ and $\phi^\varepsilon(t)$ at infinity in space, for $t > 0$, in particular if $\psi_{\text{in}}^\varepsilon$ and $\phi_{\text{in}}^\varepsilon$ do not have a trivial behavior at infinity. Of course, if $\psi_{\text{in}}^\varepsilon$ and $\phi_{\text{in}}^\varepsilon$ are analytic themselves, we have the following properties.

Corollary 4.1.9. *Assume that* $(\psi_{\text{in}}^\varepsilon, \phi_{\text{in}}^\varepsilon)$ *satisfies Assumption 1. Define* T*,* $\delta(t)$ *and* $(\zeta^\varepsilon, v^\varepsilon)$ *given by Corollary 4.1.5. If* $\psi_{\text{in}}^\varepsilon \in \mathcal{H}_{\delta_{\text{in}}}^{\ell+1}$*, then*

$$\psi^\varepsilon \in \mathcal{C}_T\mathcal{H}_\delta^{\ell+\frac{1}{2}} \cap L_T^\infty\mathcal{H}_\delta^{\ell+1} \cap L_T^2\mathcal{H}_\delta^{\ell+\frac{3}{2}}.$$

Furthermore, if we also have $\phi_{\text{in}}^\varepsilon \in \mathcal{H}_{\delta_{\text{in}}}^{\ell+2}$*, then*

$$\phi^\varepsilon \in \mathcal{C}_T\mathcal{H}_\delta^{\ell+\frac{3}{2}} \cap L_T^\infty\mathcal{H}_\delta^{\ell+2} \cap L_T^2\mathcal{H}_\delta^{\ell+\frac{5}{2}}.$$

Remark 4.1.10. From the second part of Corollary 4.1.5, we know that the behaviour at infinity of $\psi^\varepsilon(t)$ is the same as $\psi^\varepsilon_{\text{in}}$ for all t. Moreover, the behaviour of ϕ^ε is consistent with the effect of scaling for (4.1.1) (see Remark 4.1.7).

In particular, if $\psi^\varepsilon_{\text{in}} \in \mathcal{H}^\ell_{\delta_{\text{in}}}$, then $\psi^\varepsilon_{\text{in}}(x)$ (and then also $\psi^\varepsilon(t,x)$ for all $t \in [0,T]$) goes to 0 when $|x| \to \infty$, which means that $|u^\varepsilon(t,x)|$ goes to 1 (or any another positive constant if we add a constant to ψ^ε with Remark 4.1.7) when $x \to \infty$. This is therefore linked to the Gross-Pitaevskii problem.

Yet, if $\psi^\varepsilon_{\text{in}}$ goes to $-\infty$ at infinity, then so does $\psi^\varepsilon(t)$ for any $t \in [0,T]$, which means that we are close to vacuum at infinity at any time for (4.1.1) and (4.1.7). More generally, if $\psi^\varepsilon_{\text{in}}$ is bounded by above, then so are $\psi^\varepsilon(t)$ and $|u^\varepsilon(t)|$ for any $t \in [0,T]$. Moreover, in any case and for any compact subset $K \subset \mathbb{R}^d$ and any $k \in \mathbb{N}$, $\psi^\varepsilon(t)$ and $u^\varepsilon(t)$ are $\mathcal{C}^k(K)$ and all the above convergences and properties hold by substituting the analytic spaces in space by $\mathcal{C}^k(K)$ due to the fact $\mathcal{H}^\ell_\delta \subset \mathcal{C}^k(K)$ for any $\delta > 0$ and $\ell \in \mathbb{N}$.

Semiclassical limit. The convergence given in both Theorem 4.1.4 and Corollary 4.1.5 suffices to infer the convergence of quadratic observables in some way as soon as the initial data converge. Therefore, we state the following assumptions (we recall that $\tilde{D}^\varepsilon_k$ is defined in (4.1.17)):

Assumption 2. *There exists $C > 0$ such that:*

$$\tilde{D}^\varepsilon_{\ell+\frac{1}{2}} \leq C\sqrt{\varepsilon}.$$

Assumption 3. $\ell > \frac{d+1}{2}$ *and there exists $C > 0$ such that:*

$$\tilde{D}^\varepsilon_\ell \leq C\varepsilon.$$

Corollary 4.1.11. *Under Assumption 1 and Assumption 2 or 3, the position and momentum densities converge in the following sense: for any compact subset $K \subset \mathbb{R}^d$, any $k \in \mathbb{N}$ and all $T' < \frac{\delta_{\text{in}}}{M}$ such that $T' \leq T$, there holds*

$$|u^\varepsilon|^2 \underset{\varepsilon\to 0}{\longrightarrow} e^{\psi^0}, \quad \textit{and} \quad \operatorname{Im}(\varepsilon \overline{u^\varepsilon}\nabla u^\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} e^{\psi^0}\, v^0, \quad \textit{in } L^\infty((0,T'),\mathcal{C}^k(K)).$$

Furthermore, if all $\psi^\varepsilon_{\text{in}}(x)$ are uniformly bounded by above, then all $\psi^\varepsilon(t,x)$ are uniformly bounded by above in $(0,T') \times \mathbb{R}^d$ and there holds for all $k \in \mathbb{N}$

$$\left|\!\left|\!\left| |u^\varepsilon|^2 - e^{\psi^0} \right|\!\right|\!\right|_{L^\infty_{T'}H^k} + \left|\!\left|\!\left| \operatorname{Im}(\varepsilon\overline{u^\varepsilon}\nabla u^\varepsilon) - e^{\psi^0}\, v^0 \right|\!\right|\!\right|_{L^\infty_{T'}H^k} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

However, regarding convergence of the wave functions, the previous result is not sufficient. Indeed, as fast as $\phi^\varepsilon_{\text{in}}$ and $\psi^\varepsilon_{\text{in}}$ may converge as $\varepsilon \to 0$, Corollary 4.1.5 guarantees at most that $\phi^\varepsilon - \phi^0 = O(\varepsilon)$, which only ensures that $e^{\frac{\psi^\varepsilon}{2}+i\frac{\phi^\varepsilon}{\varepsilon}} - e^{\frac{\psi^0}{2}+i\frac{\phi^0}{\varepsilon}} = O(1)$ due to the rapid oscillations. In order to get a better approximation, we have to approximate ϕ^ε up to an error $o(\varepsilon)$ by adding a corrective term. For this purpose, we consider the system obtained by linearizing (4.1.9) around (ψ^0,ϕ^0), with $(\psi_{\text{in},1},\phi_{\text{in},1})$ real-valued initial data:

$$\begin{cases} \partial_t\phi_1 + v^0\cdot\nabla\phi_1 + \lambda\operatorname{Re}\psi_1 = 0, & \phi_1(0) = \phi_{\text{in},1}, \\ \partial_t\psi_1 + v^0\cdot\nabla\psi_1 + \nabla\phi_1\cdot\zeta^0 + \Delta\phi_1 = \dfrac{i}{2}\Big(\operatorname{div}\zeta^0 + 2\,\zeta^0\cdot\zeta^0\Big), & \psi_1(0) = \psi_{\text{in},1}. \end{cases} \tag{4.1.18}$$

In the same way as previously, determining ϕ^ε_1 and ψ^ε_1 is equivalent to determining $v^\varepsilon_1 = \nabla\phi^\varepsilon_1$ and $\zeta^\varepsilon_1 = \nabla\psi^\varepsilon_1$ solution to

$$\begin{cases} \partial_t v_1 + \nabla(v^0\cdot v_1) + \lambda\operatorname{Re}\zeta_1 = 0, & v_1(0) = \nabla\phi_{\text{in},1}, \\ \partial_t\zeta_1 + \nabla(v^0\cdot\zeta_1) + \nabla(v_1\cdot\zeta^0) + \nabla\operatorname{div} v_1 = \dfrac{i}{2}\Big(\nabla\operatorname{div}\zeta^0 + 2\,\nabla(\zeta^0\cdot\zeta^0)\Big), & \zeta_1(0) = \nabla\psi_{\text{in},1}, \end{cases} \tag{4.1.19}$$

along with the relations

$$\psi_1(t,x) = \psi_{\text{in},1} + \int_0^t \Big[\frac{i}{2}\Big(\operatorname{div}\zeta^0(\tau,x) + 2\,\zeta^0(\tau,x)\cdot\zeta^0(\tau,x)\Big) - v^0(\tau,x)\cdot\zeta_1(\tau,x) - v_1(\tau,x)\cdot\zeta^0(\tau,x)\Big]\,\mathrm{d}\tau,$$

$$\phi_1(t,x) = \phi_{\text{in},1} - \int_0^t \Big(v^0(\tau,x)\cdot v_1(\tau,x) + \lambda\operatorname{Re}\psi_1(\tau,x)\Big)\,\mathrm{d}\tau.$$

Remark 4.1.12. This is also equivalent to linearize (4.1.15) around (ζ^0,v^0).

Provided (ζ^0,v^0) satisfying the conclusion of Theorem 4.1.4 and $(\nabla\psi_{\text{in},1},\nabla\phi_{\text{in},1}) \in \mathcal{H}^m_{\delta_{\text{in}}} \times \mathcal{H}^{m+1}_{\delta_{\text{in}}}$ with $\frac{d-1}{2} < m \leq l-1$, we will see that the solution (ζ_1,v_1) to (4.1.19) belongs to $(L^\infty_T\mathcal{H}^m_\delta \times L^\infty_T\mathcal{H}^{m+1}_\delta) \cap (L^2_T\mathcal{H}^{m+\frac{1}{2}}_\delta \times L^2_T\mathcal{H}^{m+\frac{3}{2}}_\delta)$. Similarly, if $(\psi_{\text{in},1},\phi_{\text{in},1}) \in \mathcal{H}^{m+1}_{\delta_{\text{in}}} \times \mathcal{H}^{m+2}_{\delta_{\text{in}}}$, we will see that the solution (ψ_1,ϕ_1) to (4.1.18) belongs to $(L^\infty_T\mathcal{H}^{m+1}_\delta \times L^\infty_T\mathcal{H}^{m+2}_\delta) \cap (L^2_T\mathcal{H}^{m+\frac{3}{2}}_\delta \times L^2_T\mathcal{H}^{m+\frac{5}{2}}_\delta)$. The appearance of these correctors, and in particular regarding cases where they are trivial or not, have already been discussed in [27] in a more classical WKB framework. However, our context is a bit more particular, and we have the following:

Lemma 4.1.13. *$\phi_1(t) \equiv 0$ on $[0,T]$ if and only if $(\psi_{\text{in},1}, \phi_{\text{in},1}) \equiv (0,0)$.*

From the previous discussion, we make the following assumptions:

Assumption 4. *$\ell > \frac{d+3}{2}$ and there exists $(\psi_{\text{in},1}, \phi_{\text{in},1})$ such that*

$$r^\varepsilon_{\ell-2} \coloneqq \left\|\nabla\psi^\varepsilon_{\text{in}} - (\nabla\psi^0_{\text{in}} + \varepsilon\nabla\psi_{\text{in},1})\right\|_{\ell-2,\delta} + \left\|\nabla\phi^\varepsilon_{\text{in}} - (\nabla\phi^0_{\text{in}} + \varepsilon\nabla\phi_{\text{in},1})\right\|_{\ell-1,\delta} = o(\varepsilon) \qquad \text{as } \varepsilon \to 0.$$

Assumption 5. *$\ell > \frac{d+3}{2}$ and there exists $(\psi_{\text{in},1}, \phi_{\text{in},1})$ such that*

$$\tilde r^\varepsilon_{\ell-1} \coloneqq \left\|\psi^\varepsilon_{\text{in}} - (\psi^0_{\text{in}} + \varepsilon\psi_{\text{in},1})\right\|_{\ell-1,\delta} + \left\|\phi^\varepsilon_{\text{in}} - (\phi^0_{\text{in}} + \varepsilon\phi_{\text{in},1})\right\|_{\ell,\delta} = o(\varepsilon) \qquad \text{as } \varepsilon \to 0.$$

Our final result is stated as follows.

Theorem 4.1.14. *Under Assumption 4, there exists an ε-independent $C > 0$ such that for all $\varepsilon \in [0,1]$,*

$$\left|\!\left|\!\left|\zeta^\varepsilon - (\zeta^0 + \varepsilon\zeta_1)\right|\!\right|\!\right|_{\infty,T,\ell-2,\delta} + \left|\!\left|\!\left|\zeta^\varepsilon - (\zeta^0 + \varepsilon\zeta_1)\right|\!\right|\!\right|_{2,T,\ell-\frac{3}{2},\delta} \le C(r^\varepsilon_{\ell-2} + \varepsilon^2),$$
$$\left|\!\left|\!\left|v^\varepsilon - (v^0 + \varepsilon v_1)\right|\!\right|\!\right|_{\infty,T,\ell-1,\delta} + \left|\!\left|\!\left|v^\varepsilon - (v^0 + \varepsilon v_1)\right|\!\right|\!\right|_{2,T,\ell-\frac{1}{2},\delta} \le C(r^\varepsilon_{\ell-2} + \varepsilon^2).$$

Moreover, if Assumption 5 is satisfied, then there also holds for all $\varepsilon \in [0,1]$,

$$\left|\!\left|\!\left|\psi^\varepsilon - (\psi^0 + \varepsilon\psi_1)\right|\!\right|\!\right|_{\infty,T,\ell-1,\delta} + \left|\!\left|\!\left|\psi^\varepsilon - (\psi^0 + \varepsilon\psi_1)\right|\!\right|\!\right|_{2,T,\ell-\frac{1}{2},\delta} \le C(\tilde r^\varepsilon_{\ell-1} + \varepsilon^2),$$
$$\left|\!\left|\!\left|\phi^\varepsilon - (\phi^0 + \varepsilon\phi_1)\right|\!\right|\!\right|_{\infty,T,\ell,\delta} + \left|\!\left|\!\left|\phi^\varepsilon - (\phi^0 + \varepsilon\phi_1)\right|\!\right|\!\right|_{2,T,\ell+\frac{1}{2},\delta} \le C(\tilde r^\varepsilon_{\ell-1} + \varepsilon^2).$$

In particular, for any compact subset $K \subset \mathbb{R}^d$ and $k \in \mathbb{N}$ and $T' < \frac{\delta_{\text{in}}}{M}$ such that $T' \le T$, there holds

$$\left\|u^\varepsilon - e^{\frac{\psi^0}{2} + i\phi_1 + i\frac{\phi^0}{\varepsilon}}\right\|_{L^\infty_{T'}\mathcal{C}^k(K)} = O\Big(\frac{r^\varepsilon_1}{\varepsilon} + \varepsilon\Big) \underset{\varepsilon\to0}{\longrightarrow} 0$$

and, if $\psi^\varepsilon_{\text{in}}$ is uniformly bounded by above,

$$\left\|u^\varepsilon - e^{\frac{\psi^0}{2} + i\phi_1 + i\frac{\phi^0}{\varepsilon}}\right\|_{L^\infty_{T'}\mathcal{C}^k_b(\mathbb{R}^d)} = O\Big(\frac{r^\varepsilon_1}{\varepsilon} + \varepsilon\Big) \underset{\varepsilon\to0}{\longrightarrow} 0$$

4.1.6. Outline

In Section 4.2, we first state a toolbox lemma for the computations in analytic spaces, and then address the Cauchy theory in Theorem 4.1.4 in two steps. First, we prove the existence part thanks to a scheme defined in Section 4.2.2. Then, we show the uniqueness of this solution through similar estimates as in the existence part. Section 4.3 is devoted to the semiclassical limit, with the proof of the second part of Theorem 4.1.4. We prove the results about ψ^ε and v^ε, *i.e.* the second and third parts of Corollary 4.1.5, in Section 4.4. Section 4.5 is devoted to the semiclassical limit of the wave function: we address there Lemma 4.1.13 and Theorem 4.1.14. Last, we discuss in Section 4.6 the assumptions on the initial data, and in particular the differences from the direct assumption $(\psi^\varepsilon_{\text{in}}, \nabla\phi^\varepsilon_{\text{in}}) \in (\mathcal{H}^{\ell+1}_{\delta_{\text{in}}})^2$ for instance.

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4.2. CAUCHY PROBLEM

In this section, we prove Theorem 4.1.4. Our proof is based on an iterative scheme in a similar way as in [34] for example even though it is a little different.

4.2.1. Analytic functions

We recall that the analytic spaces have been defined in Section 4.1.5. We first recall some properties of these spaces (see [77]).

Lemma 4.2.1. *Let $\ell, \delta > 0$.*

1. *For any $\alpha \in \mathbb{N}^d$ and $f \in \mathcal{H}_\delta^{\ell+|\alpha|}$,*
$$\|\partial_x^\alpha f\|_{\ell,\delta} \leq \|f\|_{\ell+|\alpha|,\delta}.$$
More precisely, we have:
$$\|f\|_{\ell+1,\delta}^2 = \|f\|_{\ell,\delta}^2 + \sum_{|\alpha|=1} \|\partial_x^\alpha f\|_{\ell,\delta}^2. \tag{4.2.1}$$
2. *For any $s \in \mathbb{R}$, $f \in \mathcal{H}_\delta^{\ell+s}$ and $g \in \mathcal{H}_\delta^{\ell-s}$,*
$$\langle f, g\rangle_{\mathcal{H}_\delta^\ell} \leq \|f\|_{\ell+s,\delta}\|g\|_{\ell-s,\delta}.$$
3. *For any $m > \frac{d}{2}$, there exists $K^{\ell,m} > 0$ (if $\ell = m$, we will simply denote it by K^ℓ) which does not depend on $\delta > 0$ such that for any $f, g \in \mathcal{H}_\delta^{\max(m,\ell)}$,*
$$\|f \cdot g\|_{\ell,\delta} \leq \frac{1}{2}K^{\ell,m}\Big(\|f\|_{m,\delta}\|g\|_{\ell,\delta} + \|f\|_{\ell,\delta}\|g\|_{m,\delta}\Big).$$
4. *For any $f \in \mathcal{H}_\delta^\ell$, if f is scalar, then*
$$\operatorname{Re}\langle f, i\Delta f\rangle_{\ell,\delta} = 0;$$
if f is $\mathbb{R}^d$-valued, then
$$\operatorname{Re}\langle f, i\nabla \operatorname{div} f\rangle_{\ell,\delta} = 0;$$
5. *If $\ell > \frac{d}{2}$, we have a constant $C > 0$ such that for all $\delta > 0$ and all $f \in \mathcal{H}_\delta^\ell$,*
$$\|f\|_{L^\infty} \leq C\|f\|_{H^\ell} \leq C\|f\|_{\ell,\delta}.$$

Moreover, as already said in Section 4.1.5, we take time-depending $\delta(t)$ for the analytic regularity $\mathcal{H}_\delta^\ell$. $\|f\|_{\ell,\delta}^2$ for time-depending f and δ can be estimated thanks to the following result.

Lemma 4.2.2. *For a $\mathcal{C}^1$ time-dependent δ, we have:*
$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{\ell,\delta}^2 = 2\dot{\delta}\|f\|_{\ell+\frac{1}{2},\delta}^2 + 2\operatorname{Re}\langle f, \partial_t f\rangle_{\ell,\delta}.$$

The following lemma, based on the previous properties, is a toolbox for all the forthcoming analysis and estimates. For a partial proof, we refer to [34], most cases not treated in there can be treated in a similar way thanks to Lemma 4.2.1.

Lemma 4.2.3. *Let $m > \frac{d}{2} - 1$, $M > 0$, $\delta(t) = \delta_{\text{in}} - Mt$ and $T \leq \frac{\delta_{\text{in}}}{M}$. Let $(f, g) \in \mathcal{C}([0,T], \mathcal{H}_\delta^{m+\frac{1}{2}} \times \mathcal{H}_\delta^{m-\frac{1}{2}})$, and denote $\mathrm{D} \coloneqq \Delta$ if they are $\mathbb{C}$-valued or $\mathrm{D} \coloneqq \nabla \operatorname{div}$ if they are $\mathbb{C}^d$-valued. Let $(F, G) \in L^2((0,T), \mathcal{H}_\delta^{m+\frac{1}{2}} \times \mathcal{H}_\delta^{m-\frac{1}{2}})$, $\tilde{g}_1 \in L_T^2\mathcal{H}_\delta^{m+1}$, $\tilde{g}_2 \in L_T^\infty\mathcal{H}_\delta^{m+1}$ and $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$, and assume that $g \in L_T^2\mathcal{H}_\delta^{m+1}$ if $\theta_2 \neq 0$ and*
$$\partial_t f = F, \qquad f(0) \in \mathcal{H}_{\delta_{\text{in}}}^{m+1},$$
$$\partial_t g = G + i\theta_1 \mathrm{D}\, g + i\theta_2 \mathrm{D}\, \tilde{g}_1 + i\theta_3\nabla(\tilde{g}_2 \cdot \tilde{g}_2), \qquad g(0) \in \mathcal{H}_{\delta_{\text{in}}}^m.$$
Then, $t \mapsto \|f(t)\|_{m+1,\delta(t)}^2$ and $t \mapsto \|g(t)\|_{m,\delta(t)}^2$ are continuous and for all $t \in [0,T]$,
$$\mathcal{E}_{M,m+1,\delta}(f)(t) \leq \|f(0)\|_{m+1,\delta}^2 + 2|||f|||_{2,t,m+\frac{3}{2},\delta}|||F|||_{2,t,m+\frac{1}{2},\delta},$$
$$\mathcal{E}_{M,m,\delta}(g)(t) \leq \|g(0)\|_{m,\delta}^2 + 2|||g|||_{2,t,m+\frac{1}{2},\delta}|||G|||_{2,t,m-\frac{1}{2},\delta} + 2|\theta_2||||g|||_{2,t,m+1,\delta}|||\tilde{g}_1|||_{2,t,m+1,\delta} + 2T|\theta_3||||g|||_{\infty,t,m,\delta}|||\tilde{g}_2|||_{\infty,t,m+1,\delta}^2.$$
In the case $\theta_2 = 0$, the term $2|\theta_2||||g|||_{2,m+1,\delta}|||\tilde{g}_1|||_{2,m+1,\delta}$ should be understood to be zero in any case. Moreover, there holds for all $t \in [0,T]$

— *If* $F = F_1 \cdot F_2$ *with* $F_1 \in L^\infty_T \mathcal{H}^{m+\frac{1}{2}}_\delta$ *and* $F_2 \in L^2_T \mathcal{H}^{m+\frac{3}{2}}_\delta$ *and* $m > \frac{d-1}{2}$, *then*

$$|||F|||_{2,t,m+\frac{1}{2},\delta} \leq K^{m+\frac{1}{2}} |||F_1|||_{\infty,t,m+\frac{1}{2},\delta} |||F_2|||_{2,t,m+\frac{3}{2},\delta}, \tag{4.2.2}$$

where $K^{m+\frac{1}{2}}$ *is defined in Lemma 4.2.1 (part 3).*

— *If* $F = (F_1 \cdot \nabla)F_2$ *with* $F_1 \in L^\infty_T \mathcal{H}^{m+\frac{1}{2}}_\delta$ *and* $F_2 \in L^2_T \mathcal{H}^{m+\frac{3}{2}}_\delta$ *and* $m > \frac{d-1}{2}$, *then*

$$|||F|||_{2,t,m+\frac{1}{2},\delta} \leq K^{m+\frac{1}{2}} |||F_1|||_{\infty,t,m+\frac{1}{2},\delta} |||F_2|||_{2,t,m+\frac{3}{2},\delta}. \tag{4.2.3}$$

— *If* $F = (F_1 \cdot \nabla)F_2$ *with* $F_1 \in L^2_T \mathcal{H}^{m+\frac{1}{2}}_\delta$ *and* $F_2 \in L^\infty_T \mathcal{H}^{m+\frac{3}{2}}_\delta$ *and* $m > \frac{d-1}{2}$, *then*

$$|||F|||_{2,t,m+\frac{1}{2},\delta} \leq K^{m+\frac{1}{2}} |||F_1|||_{2,t,m+\frac{1}{2},\delta} |||F_2|||_{\infty,t,m+\frac{3}{2},\delta}. \tag{4.2.4}$$

— *If* $F = \theta_4 \operatorname{Re} F_1$ *with* $F_1 \in L^2_T \mathcal{H}^{m+\frac{1}{2}}_\delta$ *and* $\theta_4 \in \mathbb{R}$, *then*

$$|||F|||_{2,t,m+\frac{1}{2},\delta} \leq 2|\theta_4| |||F_1|||_{\infty,t,m+\frac{1}{2},\delta}. \tag{4.2.5}$$

— *If* $F = \nabla(F_1 \cdot F_2)$ *with* $F_1 \in L^\infty_T \mathcal{H}^{m+\frac{3}{2}}_\delta$ *and* $F_2 \in L^2_T \mathcal{H}^{m+\frac{3}{2}}_\delta$, *then*

$$|||F|||_{2,t,m+\frac{1}{2},\delta} \leq K^{m+\frac{3}{2}} |||F_1|||_{\infty,t,m+\frac{3}{2},\delta} |||F_2|||_{2,t,m+\frac{3}{2},\delta}. \tag{4.2.6}$$

— *If* $G = \nabla(G_1 \cdot G_2)$ *with* $G_1 \in L^\infty_T \mathcal{H}^{m+\frac{1}{2}}_\delta$ *and* $G_2 \in L^2_T \mathcal{H}^{m+\frac{1}{2}}_\delta$ *and* $m > \frac{d-1}{2}$, *then*

$$|||G|||_{2,t,m-\frac{1}{2},\delta} \leq K^{m+\frac{1}{2}} |||G_1|||_{\infty,m+\frac{1}{2},\delta} |||G_2|||_{2,t,m+\frac{1}{2},\delta}. \tag{4.2.7}$$

— *If* $G = \theta_5 \operatorname{D} G_1$ *with* $G_1 \in L^2_T \mathcal{H}^{m+\frac{3}{2}}_\delta$ *and* $\theta_5 \in \mathbb{C}$, *then*

$$|||G|||_{2,t,m-\frac{1}{2},\delta} \leq |\theta_5| |||G_1|||_{2,t,m+\frac{3}{2},\delta} \tag{4.2.8}$$

— *If* $G = \theta_6 \nabla(G_1 \cdot G_1)$ *with* $G_1 \in L^\infty_T \mathcal{H}^m_\delta \cap L^2_T \mathcal{H}^{m+\frac{1}{2}}_\delta$ *and* $\theta_6 \in \mathbb{C}$ *and* $m > \frac{d}{2}$, *then*

$$|||G|||_{2,t,m-\frac{1}{2},\delta} \leq |\theta_6| K^{m+\frac{1}{2},m} |||G_1|||_{\infty,t,m,\delta} |||G_1|||_{2,t,m+\frac{1}{2},\delta}. \tag{4.2.9}$$

— *If* $G = G_1 \cdot G_2$ *with* $G_1 \in L^\infty_T \mathcal{H}^{m+1}_\delta$ *and* $G_2 \in L^2_T \mathcal{H}^{m+1}_\delta$, *then*

$$|||G|||_{2,t,m-\frac{1}{2},\delta} \leq K^{m+1} |||G_1|||_{\infty,t,m+1,\delta} |||G_2|||_{2,t,m+1,\delta}. \tag{4.2.10}$$

or, if $m > \frac{d-1}{2}$,

$$|||G|||_{2,t,m-\frac{1}{2},\delta} \leq K^{m+\frac{1}{2}} |||G_1|||_{\infty,t,m+\frac{1}{2},\delta} |||G_2|||_{2,t,m+\frac{1}{2},\delta}. \tag{4.2.11}$$

— *If* $G = \theta_7 \operatorname{div} G_1$ *with* $G_1 \in L^2_T \mathcal{H}^{m+\frac{1}{2}}_\delta$ *and* $\theta_7 \in \mathbb{C}$, *then*

$$|||G|||_{2,t,m-\frac{1}{2},\delta} \leq |\theta_7| |||G_1|||_{2,t,m+\frac{1}{2},\delta} \tag{4.2.12}$$

4.2.2. Setting of the scheme

Let $\varepsilon \in [0,1]$. Set $\zeta_0^\varepsilon(t) := \nabla\psi_{\text{in}}^\varepsilon$ and $v_0^\varepsilon(t) := \nabla\phi_{\text{in}}^\varepsilon$ for all $t \geq 0$. Then, for all $k \in \mathbb{N}$, define ζ_{k+1}^ε and v_{k+1}^ε by induction as the solution to

$$\begin{cases} \partial_t v_{k+1}^\varepsilon + (v_k^\varepsilon \cdot \nabla) v_{k+1}^\varepsilon + \lambda \operatorname{Re} \zeta_k^\varepsilon = 0, & v_{k+1}^\varepsilon(0) = \nabla\phi_{\text{in}}^\varepsilon, \\ \partial_t \zeta_{k+1}^\varepsilon + \nabla\Big(v_k^\varepsilon \cdot \zeta_k^\varepsilon\Big) + \nabla \operatorname{div} v_{k+1}^\varepsilon = i\dfrac{\varepsilon}{2}\Big(\nabla \operatorname{div} \zeta_{k+1}^\varepsilon + 2\nabla(\zeta_k^\varepsilon \cdot \zeta_k^\varepsilon)\Big), & \zeta_{k+1}^\varepsilon(0) = \nabla\psi_{\text{in}}^\varepsilon. \end{cases} \tag{4.2.13}$$

The first equation is an explicit transport equation with source term and does not depend on ζ_{k+1}^ε so that v_{k+1}^ε can be defined first independently. For our case, we will show that those terms are smooth (and even analytic). Then, the second equation can be solved thanks to the Schrödinger semigroup:

$$\zeta_{k+1}^\varepsilon(t) = \nabla e^{i\frac{\varepsilon}{2}t\Delta}\psi_{\text{in}}^\varepsilon - \nabla \int_0^t e^{i\frac{\varepsilon}{2}(t-\tau)\Delta}\Big(v_k^\varepsilon(\tau) \cdot \zeta_k^\varepsilon(\tau) + \frac{1}{2}\operatorname{div} v_{k+1}^\varepsilon(\tau) - i\frac{\varepsilon}{2}\zeta_k^\varepsilon(\tau) \cdot \zeta_k^\varepsilon(\tau)\Big)\,\mathrm{d}\tau. \tag{4.2.14}$$

It is easy to see that ζ_{k+1}^ε defined by (4.2.14) satisfies (4.2.13). Indeed, define

$$\psi_{k+1}^\varepsilon(t) = e^{i\frac{\varepsilon}{2}t\Delta}\psi_{\text{in}}^\varepsilon - \int_0^t e^{i\frac{\varepsilon}{2}(t-\tau)\Delta}\Big(v_k^\varepsilon(\tau) \cdot \zeta_k^\varepsilon(\tau) + \frac{1}{2}\operatorname{div} v_{k+1}^\varepsilon(\tau) - i\frac{\varepsilon}{2}\zeta_k^\varepsilon(\tau) \cdot \zeta_k^\varepsilon(\tau)\Big)\,\mathrm{d}\tau,$$

then it easy to check that $\zeta_{k+1}^\varepsilon = \nabla\psi_{k+1}^\varepsilon$ and

$$\partial_t \psi_{k+1}^\varepsilon - i\frac{\varepsilon}{2}\Delta\psi_{k+1}^\varepsilon = -\Big(v_k^\varepsilon \cdot \zeta_k^\varepsilon + \frac{1}{2}\operatorname{div} v_{k+1}^\varepsilon - i\frac{\varepsilon}{2}\zeta_k^\varepsilon \cdot \zeta_k^\varepsilon\Big).$$

4.2.3. Well-posedness of the scheme

Fix now $\psi^\varepsilon_{\text{in}}, \phi^\varepsilon_{\text{in}}$ satisfying Assumption 1. With this assumption, our scheme is well-posed (at least locally in time).

Lemma 4.2.4. *There exists $M > 0$ and $T \in (0, \frac{\delta_{\text{in}}}{M}]$ such that, for $\delta(t) := \delta_{\text{in}} - Mt$, $(v^\varepsilon_k, \zeta^\varepsilon_k)$ is well defined and uniformly bounded in both $k \in \mathbb{N}$ and $\varepsilon \in [0,1]$ in $\mathcal{C}([0,T], \mathcal{H}^{\ell+1}_\delta \times \mathcal{H}^\ell_\delta) \cap L^2((0,T), \mathcal{H}^{\ell+\frac{3}{2}}_\delta \times \mathcal{H}^{\ell+\frac{1}{2}}_\delta)$.*

Proof. We show this result by induction. The fact that ζ^ε_0 and v^ε_0 are well defined is obviously true. Since $\partial_t v^\varepsilon_0 = \partial_t \zeta^\varepsilon_0 = 0$, Lemma 4.2.3 gives for all $t \geq 0$ with $\delta(t) = \delta_{\text{in}} - Mt$:

$$\mathcal{E}_{M,\ell,\delta}(\zeta^\varepsilon_0)(t) + \mathcal{E}_{M,\ell+1,\delta}(v^\varepsilon_0)(t) \leq \|\zeta^\varepsilon_{\text{in}}\|^2_{\ell,\delta_{\text{in}}} + \|v^\varepsilon_{\text{in}}\|^2_{\ell+1,\delta_{\text{in}}} \leq \omega_{\text{in}}. \tag{4.2.15}$$

Therefore, we have $(\zeta^\varepsilon_0, v^\varepsilon_0) \in L^\infty((0,T), \mathcal{H}^{\ell+1}_\delta \times \mathcal{H}^\ell_\delta) \cap L^2((0,T), \mathcal{H}^{\ell+\frac{3}{2}}_\delta \times \mathcal{H}^{\ell+\frac{1}{2}}_\delta)$ as long as we take $M > 0$. Now, assume that it is true for some $k \geq 0$. With this property, v^ε_{k+1} is solution of a transport equation with explicit smooth terms and is therefore well defined (thanks to characteristics). Then, ζ^ε_{k+1} is also well defined thanks to (4.2.14) along with the property of the Schrödinger semigroup in analytic spaces. Then, we use Lemma 4.2.3 with $f = v^\varepsilon_{k+1}$, $g = \zeta^\varepsilon_{k+1}$, $m = \ell$, $\theta_1 = \frac{\varepsilon}{2}$, (4.2.3), (4.2.5) with $\theta_4 = \lambda$ and (4.2.7)-(4.2.9) with $\theta_5 = 1$ and $\theta_6 = i\frac{\varepsilon}{2}$. For that, set

$$\omega^\varepsilon_k(t) := \|\zeta^\varepsilon_k\|^2_{\infty,t,\ell,\delta} + \|v^\varepsilon_k\|^2_{\infty,t,\ell+1,\delta},$$
$$\eta^\varepsilon_k(t) := \|\zeta^\varepsilon_k\|^2_{2,t,\ell+\frac{1}{2},\delta} + \|v^\varepsilon_k\|^2_{2,t,\ell+\frac{3}{2},\delta}.$$

We also use the following computations:

$$2\,|||v^\varepsilon_{k+1}|||_{2,t,\ell+\frac{3}{2},\delta}|||\zeta^\varepsilon_k|||_{2,t,\ell+\frac{1}{2},\delta} \leq |||v^\varepsilon_{k+1}|||^2_{2,t,\ell+\frac{3}{2},\delta} + |||\zeta^\varepsilon_k|||^2_{2,t,\ell+\frac{1}{2},\delta},$$

$$|||\zeta^\varepsilon_{k+1}|||_{2,t,\ell+\frac{1}{2},\delta}|||v^\varepsilon_k|||_{\infty,t,\ell+\frac{1}{2},\delta}|||\zeta^\varepsilon_k|||_{2,t,\ell+\frac{1}{2},\delta} \leq \frac{1}{2}|||v^\varepsilon_k|||^2_{\infty,t,\ell+1,\delta}|||\zeta^\varepsilon_{k+1}|||^2_{2,t,\ell+\frac{1}{2},\delta} + \frac{1}{2}|||\zeta^\varepsilon_k|||^2_{2,t,\ell+\frac{1}{2},\delta},$$

$$|||\zeta^\varepsilon_{k+1}|||_{2,t,\ell+\frac{1}{2},\delta}|||v^\varepsilon_{k+1}|||_{2,t,\ell+\frac{3}{2},\delta} \leq \frac{1}{2}\eta^\varepsilon_{k+1},$$

$$|||\zeta^\varepsilon_{k+1}|||_{2,t,\ell+\frac{1}{2},\delta}|||\zeta^\varepsilon_k|||_{\infty,t,\ell,\delta}|||\zeta^\varepsilon_k|||_{2,t,\ell+\frac{1}{2},\delta} \leq \frac{1}{2}|||\zeta^\varepsilon_k|||^2_{\infty,t,\ell,\delta}|||\zeta^\varepsilon_{k+1}|||^2_{2,t,\ell+\frac{1}{2},\delta} + \frac{1}{2}|||\zeta^\varepsilon_k|||^2_{2,t,\ell+\frac{1}{2},\delta}.$$

Therefore, for all $\varepsilon \leq 1$ and $t \geq 0$ such that $\delta(t) \geq 0$ and using Assumption 1,

$$\begin{aligned}\mathcal{E}_{M,\ell,\delta}(\zeta^\varepsilon_{k+1})(t) + \mathcal{E}_{M,\ell+1,\delta}(v^\varepsilon_{k+1})(t) \leq \omega_{\text{in}} + (2C_\ell(\sqrt{\omega^\varepsilon_k(t)} + \omega^\varepsilon_k(t)) + 2|\lambda| + 1)\eta^\varepsilon_{k+1}(t) \\ + 2(|\lambda| + C_\ell)\,|||\zeta^\varepsilon_k|||^2_{2,t,\ell+\frac{1}{2},\delta},\end{aligned} \tag{4.2.16}$$

for some $C_\ell > 0$ depending only on ℓ. Set

$$M_1 := C_\ell(\sqrt{2\omega_{\text{in}}} + 2\omega_{\text{in}}) + |\lambda| + \frac{1}{2},$$
$$M_2 := |\lambda| + C_\ell,$$
$$M := M_1 + 2M_2.$$

Moreover, take $\delta(t) = \delta_{\text{in}} - Mt$ and set $T = M^{-1}\delta_{\text{in}}$ so that $\delta(t) \geq 0$ for all $t \in [0,T]$. From these estimates and definitions, we can prove that the scheme is uniformly bounded thanks to the following Lemma.

Lemma 4.2.5. *For all $\varepsilon \in [0,1]$, $t \in [0,T]$ and $k \in \mathbb{N}$, there holds*

$$\mathcal{E}_{2M_2,\ell,\delta}(\zeta^\varepsilon_k)(t) + \mathcal{E}_{2M_2,\ell+1,\delta}(v^\varepsilon_k)(t) \leq 2\omega_{\text{in}}.$$

The proof is therefore complete. □

Proof of Lemma 4.2.5. We prove this lemma by induction on k. The estimate for $k = 0$ follows from (4.2.15) and the facts that

$$-M \leq -2M_2.$$

Now, for $k \in \mathbb{N}$, assuming that the estimate holds at rank k, we have in particular the fact that $\omega^\varepsilon_k(t) \leq 2\omega_{\text{in}}$ for all $t \in [0,T]$, so that (4.2.16) becomes

$$\mathcal{E}_{2M_2,\ell,\delta}(\zeta^\varepsilon_{k+1})(t) + \mathcal{E}_{2M_2,\ell+1,\delta}(v^\varepsilon_{k+1})(t) \leq \omega_{\text{in}} + 2M_2\eta^\varepsilon_k(t).$$

Using again the property at rank k, we have for all $t \in [0,T]$

$$2M_2\eta^\varepsilon_k(t) \leq \omega_{\text{in}},$$

and thus the property at rank $k+1$ is proved. □

4.2.4. Convergence of the scheme

We proved that the scheme is well defined. We now need to show that this scheme converges as $k \to \infty$ in order to get a solution to (4.1.15) from this limit.

Lemma 4.2.6. *Up to taking a larger $M > 0$ and a smaller $T > 0$, for any $\varepsilon \in [0,1]$, $(\zeta_k^\varepsilon, v_k^\varepsilon)_k$ is a Cauchy sequence in $\mathcal{C}([0,T], \mathcal{H}_\delta^{\ell-\frac{1}{2}} \times \mathcal{H}_\delta^{\ell+\frac{1}{2}}) \cap L^2((0,T), \mathcal{H}_\delta^{\ell} \times \mathcal{H}_\delta^{\ell+1})$.*

Proof. We know that $(\zeta_k^\varepsilon, v_k^\varepsilon)$ is uniformly bounded in $L^\infty((0,T), \mathcal{H}_\delta^{\ell} \times \mathcal{H}_\delta^{\ell+1}) \cap L^2((0,T), \mathcal{H}_\delta^{\ell+\frac{1}{2}} \times \mathcal{H}_\delta^{\ell+\frac{1}{2}})$ with Lemma 4.2.4. Set $Z_{k+1}^\varepsilon := \zeta_{k+1}^\varepsilon - \zeta_k^\varepsilon$ and $V_{k+1}^\varepsilon := v_{k+1}^\varepsilon - v_k^\varepsilon$ for $k \in \mathbb{N}$. Then, we obtain for $k \geq 1$:

$$\left\{\begin{array}{l} \partial_t V_{k+1}^\varepsilon + (v_k^\varepsilon \cdot \nabla) V_{k+1}^\varepsilon + (V_k^\varepsilon \cdot \nabla) v_k^\varepsilon + \lambda \operatorname{Re} Z_k^\varepsilon = 0, \\ \partial_t Z_{k+1}^\varepsilon + \nabla\Big(V_k^\varepsilon \cdot \zeta_k^\varepsilon\Big) + \nabla\Big(v_{k-1}^\varepsilon \cdot Z_k^\varepsilon\Big) + \nabla \operatorname{div} V_{k+1}^\varepsilon = i\dfrac{\varepsilon}{2}\Big(\nabla \operatorname{div} Z_{k+1}^\varepsilon + 2\nabla(Z_k^\varepsilon \cdot \zeta_k^\varepsilon) + 2\nabla(\zeta_{k-1}^\varepsilon \cdot Z_k^\varepsilon)\Big), \end{array}\right.$$

with zero initial data. Set

$$N_k^\varepsilon(t) := |||Z_k^\varepsilon|||_{2,t,\ell,\delta}^2 + |||V_k^\varepsilon|||_{2,t,\ell+1,\delta}^2.$$

From the previous system and Lemma 4.2.3 with $m = \ell - \frac{1}{2}$, (4.2.3)-(4.2.5), (4.2.8) and three times (4.2.7) in a similar way as previously, along with Lemma 4.2.5 and the following computations:

$$\Big|\Big|\Big|V_{k+1}^\varepsilon\Big|\Big|\Big|_{2,t,\ell+1,\delta} |||V_k^\varepsilon|||_{2,t,\ell,\delta} |||v_k^\varepsilon|||_{\infty,t,\ell+1,\delta} \leq \omega_{\text{in}} \Big|\Big|\Big|V_{k+1}^\varepsilon\Big|\Big|\Big|_{2,t,\ell+1,\delta}^2 + \frac{1}{2} |||V_k^\varepsilon|||_{2,t,\ell,\delta}^2,$$

$$\Big|\Big|\Big|Z_{k+1}^\varepsilon\Big|\Big|\Big|_{2,t,\ell,\delta} |||V_k^\varepsilon|||_{2,t,\ell,\delta} |||\zeta_k^\varepsilon|||_{\infty,t,\ell,\delta} \leq \omega_{\text{in}} \Big|\Big|\Big|Z_{k+1}^\varepsilon\Big|\Big|\Big|_{2,t,\ell,\delta}^2 + \frac{1}{2} |||V_k^\varepsilon|||_{2,t,\ell,\delta}^2,$$

$$\Big|\Big|\Big|Z_{k+1}^\varepsilon\Big|\Big|\Big|_{2,t,\ell,\delta} |||Z_k^\varepsilon|||_{2,t,\ell,\delta} \sqrt{2\omega_{\text{in}}} \leq \omega_{\text{in}} \Big|\Big|\Big|Z_{k+1}^\varepsilon\Big|\Big|\Big|_{2,t,\ell,\delta}^2 + \frac{1}{2} |||Z_k^\varepsilon|||_{2,t,\ell,\delta}^2,$$

we get:

$$\begin{aligned} \mathcal{E}_{M,\ell-\frac{1}{2},\delta}(Z_{k+1}^\varepsilon)(t) + \mathcal{E}_{M,\ell+\frac{1}{2},\delta}(V_{k+1}^\varepsilon)(t) &\leq (2K^\ell \sqrt{2\omega_{\text{in}}} + 2K^\ell \omega_{\text{in}}(2+\varepsilon) + 1 + 2|\lambda|) N_{k+1}^\varepsilon(t) \\ &\quad + 2K^\ell \|V_k^\varepsilon\|_{2,\ell,\delta}^2 + (2|\lambda| + K^\ell(1+2\varepsilon)) |||Z_k^\varepsilon|||_{2,t,\ell,\delta}^2. \end{aligned}$$

Therefore, for $\varepsilon \leq 1$ and up to taking a slightly larger C_ℓ in M_1, we get

$$\mathcal{E}_{2M_2,\ell-\frac{1}{2},\delta}(Z_{k+1}^\varepsilon)(t) + \mathcal{E}_{2M_2,\ell+\frac{1}{2},\delta}(V_{k+1}^\varepsilon)(t) \leq 2M_2 N_k^\varepsilon(t). \tag{4.2.17}$$

From this estimate, we can prove a uniform estimate.

Lemma 4.2.7. *There holds for all $k \in \mathbb{N}$ and $t \in [0,T]$*

$$\mathcal{I}_{k+1}^\varepsilon(t) := \mathcal{E}_{2M_2,\ell-\frac{1}{2},\delta}(Z_{k+1}^\varepsilon)(t) + \mathcal{E}_{2M_2,\ell+\frac{1}{2},\delta}(V_{k+1}^\varepsilon)(t) \leq \omega_{\text{in}} 2^{-k+2}.$$

We first finish the proof of Lemma 4.2.6 before proving this Lemma. For any $j > k \geq 1$, there holds with Lemma 4.2.7:

$$\begin{aligned} \Big|\Big|\Big|\zeta_j^\varepsilon - \zeta_k^\varepsilon\Big|\Big|\Big|_{\infty,T,\ell-\frac{1}{2},\delta} &= \left|\left|\left|\sum_{m=k}^{j-1} Z_{m+1}^\varepsilon\right|\right|\right|_{\infty,T,\ell-\frac{1}{2},\delta} \\ &\leq \sum_{m=k}^{j-1} \Big|\Big|\Big|Z_{m+1}^\varepsilon(t)\Big|\Big|\Big|_{\infty,T,\ell-\frac{1}{2},\delta} \\ &\leq \sum_{m=k}^{j-1} \sqrt{\omega_{\text{in}}}(\sqrt{2})^{-m+2} \\ &\leq C\sqrt{2}^{-k}. \end{aligned}$$

Therefore $(\zeta_k^\varepsilon)_k$ is a Cauchy sequence in $\mathcal{C}([0,T], \mathcal{H}_\delta^{\ell-\frac{1}{2}})$ since it is continuous with values in the same space. The same kind of estimate can be proved for v_k^ε and for the $L^2((0,T), \mathcal{H}_\delta^{\ell} \times \mathcal{H}_\delta^{\ell+1})$ norm in the same way. □

Proof of Lemma 4.2.7. (4.2.17) leads to

$$\mathcal{I}_{k+1}^{\varepsilon} \leq \frac{1}{2}\mathcal{I}_{k}^{\varepsilon}.$$

Thus, we get

$$\mathcal{I}_{k+1}^{\varepsilon} \leq 2^{-k}\mathcal{I}_{1}^{\varepsilon}.$$

Moreover,

$$\|\!|Z_1^\varepsilon|\!\|_{\infty,T,\ell-\frac{1}{2},\delta} + \|\!|V_1^\varepsilon|\!\|_{\infty,T,\ell+\frac{1}{2},\delta} \leq \omega_1^\varepsilon(t) + \omega_0^\varepsilon(t),$$

and

$$2M_2 N_1^\varepsilon(t) \leq 2M_2 \int_0^t \Big(\|\zeta_1^\varepsilon(\tau)\|_{\ell,\delta}^2 + \|v_1^\varepsilon(\tau)\|_{\ell+1,\delta}^2 \Big)\,\mathrm{d}\tau + 2M_2 \int_0^t \Big(\|\zeta_0^\varepsilon(\tau)\|_{\ell,\delta}^2 + \|v_0^\varepsilon(\tau)\|_{\ell+1,\delta}^2 \Big)\,\mathrm{d}\tau.$$

Therefore, from Lemma 4.2.5, we can deduce that

$$\mathcal{I}_1^\varepsilon \leq 4\omega_{\text{in}},$$

and then the conclusion. □

From Lemma 4.2.6 along with Lemma 4.2.5, we can complete the proof of the existence of a solution to (4.1.15) and the uniform estimates in ε.

Corollary 4.2.8. *$(\zeta_k^\varepsilon, v_k^\varepsilon)_k$ converges in $\mathcal{C}([0,T], \mathcal{H}_\delta^{\ell-\frac{1}{2}} \times \mathcal{H}_\delta^{\ell+\frac{1}{2}}) \cap L^2((0,T), \mathcal{H}_\delta^{\ell} \times \mathcal{H}_\delta^{\ell+1})$ to some $(\zeta^\varepsilon, v^\varepsilon)$ solution to* (4.1.15)*, which also satisfies*

$$\|\zeta^\varepsilon(t)\|_{\ell,\delta}^2 + \|v^\varepsilon(t)\|_{\ell+1,\delta}^2 + 4M_2 \int_0^t \Big(\|\zeta^\varepsilon(\tau)\|_{\ell+\frac{1}{2},\delta}^2 + \|v^\varepsilon(\tau)\|_{\ell+\frac{3}{2},\delta}^2 \Big)\,\mathrm{d}\tau \leq 2\omega_{\text{in}}.$$

4.2.5. Uniqueness of the solution

We have just proved the existence of a solution to (4.1.15). We now prove the uniqueness of this solution thanks to similar estimates.

Lemma 4.2.9. *The solution $(\zeta^\varepsilon, v^\varepsilon)$ to* (4.1.15) *in $\mathcal{C}([0,T], \mathcal{H}_\delta^{\ell-\frac{1}{2}} \times \mathcal{H}_\delta^{\ell+\frac{1}{2}}) \cap L^\infty((0,T), \mathcal{H}_\delta^{\ell} \times \mathcal{H}_\delta^{\ell+1}) \cap L^2((0,T), \mathcal{H}_\delta^{\ell+\frac{1}{2}} \times \mathcal{H}_\delta^{\ell+\frac{3}{2}})$ is unique.*

Proof. Let $(\tilde{\zeta}^\varepsilon, \tilde{v}^\varepsilon)$ be another solution to (4.1.15) in the space $L^\infty((0,T), \mathcal{H}_\delta^{\ell} \times \mathcal{H}_\delta^{\ell+1}) \cap \mathcal{C}([0,T], \mathcal{H}_\delta^{\ell-1} \times \mathcal{H}_\delta^{\ell}) \cap \cap L^2((0,T), \mathcal{H}_\delta^{\ell+\frac{1}{2}} \times \mathcal{H}_\delta^{\ell+\frac{3}{2}})$. In a similar way as in Section 4.2.3, defining

$$\tilde{\omega}^\varepsilon(t) := \left\|\tilde{\zeta}^\varepsilon\right\|_{\infty,t,\ell,\delta}^2 + \|\tilde{v}^\varepsilon\|_{\infty,t,\ell+1,\delta}^2,$$
$$\tilde{\eta}^\varepsilon(t) := \left\|\tilde{\zeta}^\varepsilon\right\|_{2,t,\ell+\frac{1}{2},\delta}^2 + \|\tilde{v}^\varepsilon\|_{2,t,\ell+\frac{3}{2},\delta}^2,$$

there holds for all $t \in [0,T]$

$$\mathcal{E}_{M,\ell,\delta}(\tilde{\zeta}^\varepsilon)(t) + \mathcal{E}_{M,\ell+1,\delta}(\tilde{v}^\varepsilon)(t) \leq \omega_{\text{in}} + (2C_\ell(\sqrt{\tilde{\omega}^\varepsilon} + \tilde{\omega}^\varepsilon) + 2|\lambda| + 1 + 2C_\ell)\,\tilde{\eta}^\varepsilon.$$

Moreover, also from Lemma 4.2.3, $\left\|\tilde{\zeta}^\varepsilon(t)\right\|_{\ell,\delta}^2 + \|\tilde{v}^\varepsilon(t)\|_{\ell+1,\delta}^2$ is continuous in time, and then so is $\tilde{\omega}^\varepsilon(t)$. Therefore, the same kind of argument as already used along with a bootstrap property yields

$$\mathcal{E}_{|\lambda|,\ell,\delta}(\tilde{\zeta}^\varepsilon)(t) + \mathcal{E}_{|\lambda|,\ell+1,\delta}(\tilde{v}^\varepsilon)(t) \leq \omega_{\text{in}}. \tag{4.2.18}$$

Set now $\tilde{Z}^\varepsilon := \tilde{\zeta}^\varepsilon - \zeta^\varepsilon$ and $\tilde{V}^\varepsilon := \tilde{v}^\varepsilon - v^\varepsilon$. Then, we obtain:

$$\begin{cases} \partial_t \tilde{V}^\varepsilon + (\tilde{v}^\varepsilon \cdot \nabla)\tilde{V}^\varepsilon + (\tilde{V}^\varepsilon \cdot \nabla)v^\varepsilon + \lambda\,\mathrm{Re}\,\tilde{Z}^\varepsilon = 0, \\ \partial_t \tilde{Z}^\varepsilon + \nabla\Big(\tilde{V}^\varepsilon \cdot \tilde{\zeta}^\varepsilon\Big) + \nabla\Big(v^\varepsilon \cdot \tilde{Z}^\varepsilon\Big) + \nabla\,\mathrm{div}\,\tilde{V}^\varepsilon = i\dfrac{\varepsilon}{2}\Big(\nabla\,\mathrm{div}\,\tilde{Z}^\varepsilon + 2\,\nabla(\tilde{Z}^\varepsilon \cdot \tilde{\zeta}^\varepsilon) + 2\,\nabla(\zeta^\varepsilon \cdot \tilde{Z}^\varepsilon)\Big). \end{cases}$$

In the same way as in Section 4.2.4, defining

$$\tilde{\Omega}^\varepsilon(t) := \left\|\tilde{Z}^\varepsilon\right\|_{\infty,t,\ell-\frac{1}{2},\delta}^2 + \left\|\tilde{V}^\varepsilon\right\|_{\infty,t,\ell+\frac{1}{2},\delta}^2,$$
$$\tilde{N}^\varepsilon(t) := \left\|\tilde{Z}^\varepsilon\right\|_{2,t,\ell,\delta}^2 + \left\|\tilde{V}^\varepsilon\right\|_{2,t,\ell+1,\delta}^2,$$

Lemma 4.2.3 with (4.2.3)-(4.2.5) and (4.2.7)-(4.2.8) yields for all $t \in [0, T]$

$$\mathcal{E}_{M,\ell,\delta}(\tilde{Z}^\varepsilon)(t) + \mathcal{E}_{M,\ell+1,\delta}(\tilde{V}^\varepsilon)(t) \leq (2K^\ell\sqrt{\omega_{\rm in}} + K^\ell\omega_{\rm in}(2+\varepsilon) + 1 + 2|\lambda|)\tilde{N}^\varepsilon + (2|\lambda| + K^\ell(1+2\varepsilon))\,T\,\tilde{\Omega}^\varepsilon.$$

From the definition of M, we get

$$\tilde{\Omega}^\varepsilon(t) \leq (2|\lambda| + K^\ell(1+2\varepsilon))\,T\,\tilde{\Omega}^\varepsilon(t),$$

which gives the conclusion as soon as T is small enough so that $(2|\lambda| + 3K^\ell)\,T < 1$. This gives local uniqueness, which is sufficient to prove it even if T is larger. □

Remark 4.2.10. In particular, the previous computation (4.2.18) and the uniqueness property give another estimate, slightly better than that of Lemma 4.2.5 for the $L^\infty([0,T], \mathcal{H}^\ell_\delta \times \mathcal{H}^{\ell+1}_\delta)$ norm, given in the next lemma.

Lemma 4.2.11. *For all $\varepsilon \in [0,1]$, there holds for all $t \in [0,T]$*

$$\mathcal{E}_{|\lambda|,\ell,\delta}(\tilde{\zeta}^\varepsilon)(t) + \mathcal{E}_{|\lambda|,\ell+1,\delta}(\tilde{v}^\varepsilon)(t) \leq \omega_{\rm in}.$$

4.3. Semiclassical limit

We now address the semiclassical limit $\varepsilon \to 0$ in $(\zeta^\varepsilon, v^\varepsilon)$ variables, *i.e.* the proof of the second part of Theorem 4.1.4. For this, we set $Z^\varepsilon := \zeta^\varepsilon - \zeta^0$ and $V^\varepsilon := v^\varepsilon - v^0$. Using (4.1.5), they satisfy

$$\begin{cases} \partial_t V^\varepsilon + \dfrac{1}{2}\nabla(V^\varepsilon \cdot v^\varepsilon) + \dfrac{1}{2}\nabla(v^0 \cdot V^\varepsilon) + \lambda\,\mathrm{Re}\,Z^\varepsilon = 0, \qquad V^\varepsilon(0) = \nabla v^\varepsilon_{\rm in} - \nabla v^0_{\rm in} \\ \partial_t Z^\varepsilon + \nabla\Big(V^\varepsilon \cdot \zeta^\varepsilon\Big) + \nabla\Big(v^0 \cdot Z^\varepsilon\Big) + \nabla\,\mathrm{div}\,V^\varepsilon = i\dfrac{\varepsilon}{2}\Big(\nabla\,\mathrm{div}\,\zeta^\varepsilon + 2\,\nabla(\zeta^\varepsilon\cdot\zeta^\varepsilon)\Big), \quad Z^\varepsilon(0) = \nabla\psi^\varepsilon_{\rm in} - \nabla\psi^0_{\rm in}. \end{cases} \tag{4.3.1}$$

4.3.1. First case

Set

$$\Omega^\varepsilon := \|Z^\varepsilon\|^2_{\infty,t,\ell-\frac{1}{2},\delta} + \|V^\varepsilon\|^2_{\infty,t,\ell+\frac{1}{2},\delta}.$$

By applying Lemma 4.2.3 with $m = \ell - \frac{1}{2}$, $\theta_1 = 0$, $\theta_3 = 2\theta_2 = i\varepsilon$, $\tilde{g}_1 = \tilde{g}_2 = \zeta^\varepsilon$ and (4.2.5)-(4.2.8), using Lemma 4.2.11 and the following computations:

$$\begin{gathered} 2\,\|V^\varepsilon\|_{2,t,\ell+1,\delta}\|Z^\varepsilon\|_{2,t,\ell,\delta} \leq \|V^\varepsilon\|^2_{2,t,\ell+1,\delta} + \|Z^\varepsilon\|^2_{2,t,\ell,\delta}, \\ \|Z^\varepsilon\|_{2,t,\ell,\delta}\|V^\varepsilon\|_{2,t,\ell,\delta}\|\zeta^\varepsilon\|_{\infty,t,\ell,\delta} \leq \frac{T}{2}\Omega^\varepsilon + \frac{1}{2}\omega_{\rm in}\|Z^\varepsilon\|^2_{\infty,t,\ell,\delta}, \\ \left\|v^0\right\|_{\infty,t,\ell,\delta}\|Z^\varepsilon\|^2_{2,t,\ell,\delta} \leq \sqrt{\omega_{\rm in}}\|Z^\varepsilon\|^2_{2,t,\ell,\delta}, \\ \|Z^\varepsilon\|_{2,t,\ell+\frac{1}{2},\delta} \leq \|\zeta^\varepsilon\|_{2,t,\ell+\frac{1}{2},\delta} + \left\|\zeta^0\right\|_{2,t,\ell+\frac{1}{2},\delta} \leq \sqrt{\frac{2\omega_{\rm in}}{|\lambda|}}, \end{gathered}$$

we get for all $\varepsilon \leq 1$ and $t \in [0,T]$:

$$\mathcal{E}_{2M_2,\ell-\frac{1}{2},\delta}(Z^\varepsilon)(t) + \mathcal{E}_{2M_2,\ell+\frac{1}{2},\delta}(V^\varepsilon)(t) \leq (D^\varepsilon_{\ell-\frac{1}{2}})^2 + K^\ell T\Omega^\varepsilon(t) + \varepsilon\Big(\sqrt{2}K^\ell\omega_{\rm in}^{\frac{3}{2}} + \frac{\omega_{\rm in}}{|\lambda|}\Big). \tag{4.3.2}$$

Thus, up to taking for instance $T < \frac{K^\ell}{2}$, we obtain

$$\mathcal{E}_{M_2,\ell-\frac{1}{2},\delta}(Z^\varepsilon)(t) + \mathcal{E}_{M_2,\ell+\frac{1}{2},\delta}(V^\varepsilon)(t) \leq C\Big((D^\varepsilon_{\ell-\frac{1}{2}})^2 + \varepsilon\Big).$$

This gives the conclusion for the first case.

4.3.2. Case $\ell > \frac{d+1}{2}$

Set now

$$\Omega^\varepsilon(t) := \|Z^\varepsilon\|^2_{\infty,t,\ell-1,\delta} + \|V^\varepsilon\|^2_{\infty,t,\ell,\delta}.$$

Here, we use Lemma 4.2.3 in a similar way as previously. However, we treat the term $i\frac{\varepsilon}{2}\nabla\operatorname{div}\zeta^\varepsilon$ with (4.2.8) instead of with $\tilde g_1$, and the term $i\varepsilon\,\nabla(\zeta^\varepsilon\cdot\zeta^\varepsilon)$ with $\tilde g_2$ instead of with (4.2.9). Then, we also use the following computations:

$$\varepsilon\|Z^\varepsilon\|_{2,t,\ell-\frac12,\delta}\|\zeta^\varepsilon\|_{2,t,\ell+\frac12,\delta}\le C_\ell\,\omega_{\text{in}}\,\|Z^\varepsilon\|^2_{2,t,\ell-\frac12,\delta}+\varepsilon^2\frac{1}{8|\lambda|C_\ell},$$
$$\varepsilon\|Z^\varepsilon\|_{2,t,\ell-1,\delta}\|\zeta^\varepsilon\|^2_{\infty,t,\ell,\delta}\le\frac12\|Z^\varepsilon\|^2_{\infty,t,\ell-1,\delta}+\frac{\varepsilon^2}{2}T\omega_{\text{in}}^4.$$

Thus, we get

$$\mathcal{E}_{2M_2,\ell-1,\delta}(Z^\varepsilon)(t)+\mathcal{E}_{2M_2,\ell,\delta}(V^\varepsilon)(t)\le(D^\varepsilon_{\ell-1})^2+\frac12\Omega^\varepsilon(t)+\frac{\varepsilon^2}{4}\Big(T^2\,K^\ell\omega_{\text{in}}^4+\frac{1}{2|\lambda|\,C_\ell}\Big),$$

which gives the conclusion for the second statement of the second part of Theorem 4.1.4.

Remark 4.3.1. The fact that we cannot recover the $O(\varepsilon^2)$ in the first part comes from the term $\nabla\operatorname{div}\zeta^\varepsilon$. Indeed, the highest regularity for which we have an estimate for ζ^ε is the $\mathcal{H}_\delta^{\ell+\frac12}$, which is L^2 in time. Therefore, when one wants to estimate $\left|\langle Z^\varepsilon,\nabla\operatorname{div}\zeta^\varepsilon\rangle_{\ell-\frac12,\delta}\right|$ for Lemma 4.2.2 (or Lemma 4.2.3), since we cannot go further $\ell+\frac12$ in the norm of ζ^ε, there must be at least $\ell+\frac12$ for the norm of Z^ε, which is not very optimal for this estimate: we would want at most ℓ. This problem does not occur for the second case because we estimate the $(\ell-1,\delta)$ scalar product: it allows an extra notch backwards for the regularity of Z^ε, which is sufficient for falling into a better framework for our estimates.

4.4. PROPERTIES ON ψ^ε AND ϕ^ε

In this section, we prove the second and third parts of Corollary 4.1.5.

4.4.1. Behavior at infinity

Analyticity of $\partial_t\psi^\varepsilon$. The first part of this proof is a result of analyticity of $\partial_t\psi^\varepsilon$.

Lemma 4.4.1. *There holds $\partial_t\psi^\varepsilon\in L^2_T\mathcal{H}_\delta^{\ell-\frac12}$.*

Proof. We know from (4.1.11) that

$$\partial_t\psi^\varepsilon+v^\varepsilon\cdot\zeta^\varepsilon+\operatorname{div}v^\varepsilon=i\frac{\varepsilon}{2}\Big(\operatorname{div}\zeta^\varepsilon+2\,\zeta^\varepsilon\cdot\zeta^\varepsilon\Big).$$

Moreover, using Lemma 4.2.11, there holds

$$\|v^\varepsilon\cdot\zeta^\varepsilon\|_{\ell,\delta}\le K^\ell\|v^\varepsilon\|_{\ell,\delta}\|\zeta^\varepsilon\|_{\ell,\delta}\le K^\ell\omega_{\text{in}},$$
$$\|\operatorname{div}v^\varepsilon\|_{\ell,\delta}\le\|v^\varepsilon\|_{\ell+1,\delta}\le\sqrt{\omega_{\text{in}}},$$
$$\|\zeta^\varepsilon\cdot\zeta^\varepsilon\|_{\ell,\delta}\le K^\ell\|\zeta^\varepsilon\|^2_{\ell,\delta}\le K^\ell\omega_{\text{in}},$$
$$\|\operatorname{div}\zeta^\varepsilon\|_{\ell-\frac12}\le\|\zeta^\varepsilon\|_{\ell+\frac12,\delta}.$$

The latter is L^2 in time (in $(0,T)$) by using also Lemma 4.2.11. Therefore, the conclusion easily follows. □

ψ^ε case. Now, we prove the case of ψ^ε. Setting $\Psi^\varepsilon:=\psi^\varepsilon-\psi^\varepsilon_{\text{in}}$, Lemma 4.4.1 (along with the facts that $\Psi^\varepsilon(0)=0$ and $\partial_t\Psi^\varepsilon=\partial_t\psi^\varepsilon$) yields

$$\Psi^\varepsilon\in H^1((0,T),\mathcal{H}_\delta^{\ell-\frac12}).$$

In particular, $\psi^\varepsilon-\psi^\varepsilon_{\text{in}}\in\mathcal{C}([0,T],\mathcal{H}_\delta^{\ell-\frac12})$. But we also have $\nabla\psi^\varepsilon\in\mathcal{C}([0,T],\mathcal{H}_\delta^{\ell-\frac12})$, and it is of course also the same for $\nabla\psi^\varepsilon_{\text{in}}$. Therefore, by (4.2.1), we obtain

$$\psi^\varepsilon-\psi^\varepsilon_{\text{in}}\in\mathcal{C}_T\mathcal{H}_\delta^{\ell+\frac12}.$$

Moreover, Lemma 4.2.3 gives for all $t\in[0,T]$

$$\mathcal{E}_{M,\ell,\delta}(\Psi^\varepsilon)(t)\le\|\Psi^\varepsilon\|_{2,t,\ell+\frac12,\delta}\|\partial_t\Psi^\varepsilon\|_{2,t,\ell-\frac12,\delta}.$$

Applying Lemma 4.4.1 once again yields

$$\Psi^\varepsilon\in L^\infty_T\mathcal{H}_\delta^{\ell+1}\cap L^2_T\mathcal{H}_\delta^{\ell+\frac32}.$$

Thus, we get (4.1.5).

ϕ^ε *case.* In a similar way, we now prove an analyticity result for $\partial_t \Phi^\varepsilon$ where

$$\Phi^\varepsilon := \phi^\varepsilon - \phi^\varepsilon_{\text{in}} - \lambda t \psi^\varepsilon_{\text{in}}.$$

Lemma 4.4.2. *There holds* $\partial_t \Phi^\varepsilon \in L^\infty_T \mathcal{H}^{\ell+1}_\delta \cap L^2_T \mathcal{H}^{\ell+\frac{3}{2}}_\delta$.

Proof. We know that

$$\partial_t \phi^\varepsilon + \frac{1}{2}|v^\varepsilon|^2 + \lambda \operatorname{Re} \psi^\varepsilon = 0.$$

Therefore, we have

$$\partial_t \Phi^\varepsilon + \frac{1}{2}|v^\varepsilon|^2 + \lambda \operatorname{Re} \Psi^\varepsilon = 0.$$

From the previous result and from the fact that

$$\left\| |v^\varepsilon|^2 \right\|_{\ell+\frac{3}{2},\delta} \leq K^{\ell+\frac{3}{2},\ell+1} \|v^\varepsilon\|_{\ell+\frac{3}{2},\delta} \|v^\varepsilon\|_{\ell+1,\delta} \leq K^{\ell+\frac{3}{2},\ell+1} \sqrt{\omega_{\text{in}}} \|v^\varepsilon\|_{\ell+\frac{3}{2},\delta},$$

along with Lemma 4.2.11, this yields the conclusion. □

The proof of the second statement of the second part of Corollary 4.1.5 is then similar as in Section 4.4.1.

4.4.2. Semiclassical limit

Now, we prove the results of the semiclassical limit for ψ^ε and ϕ^ε. Set

$$P^\varepsilon = \psi^\varepsilon - \psi^0, \qquad Q^\varepsilon = \phi^\varepsilon - \phi^0.$$

Then, there holds

$$\partial_t P^\varepsilon + V^\varepsilon \cdot \zeta^\varepsilon + v^0 \cdot Z^\varepsilon + \operatorname{div} V^\varepsilon = i\frac{\varepsilon}{2}\Big(\operatorname{div} \zeta^\varepsilon + 2\,\zeta^\varepsilon \cdot \zeta^\varepsilon\Big),$$
$$\partial_t Q^\varepsilon + \frac{1}{2} V^\varepsilon \cdot v^\varepsilon + \frac{1}{2} v^0 \cdot V^\varepsilon + \lambda \operatorname{Re} P^\varepsilon = 0.$$

First case. By applying Lemma 4.2.3, (4.2.11) and (4.2.12) with $m = \ell - \frac{1}{2}$ and $g = P^\varepsilon$, and using the second part of Theorem 4.1.4 and Lemma 4.2.11, we obtain for all $t \in [0, T]$

$$\mathcal{E}_{M,\ell-\frac{1}{2},\delta}(P^\varepsilon)(t) \leq \|P^\varepsilon(0)\|^2_{\ell-\frac{1}{2},\delta_{\text{in}}} + C\left[\varepsilon + \Big(\varepsilon + (D^\varepsilon_{\ell-\frac{1}{2}})^2\Big)^{\frac{1}{2}}\right] \|P^\varepsilon\|_{2,t,\ell,\delta},$$

for some constant $C > 0$. Therefore we get for all $\varepsilon \in [0, 1]$

$$\mathcal{E}_{2M_2,\ell-\frac{1}{2},\delta}(P^\varepsilon)(t) \leq \|P^\varepsilon(0)\|^2_{\ell-\frac{1}{2},\delta_{\text{in}}} + C\Big(\varepsilon + (D^\varepsilon_{\ell-\frac{1}{2}})^2\Big).$$

Then, for Q^ε, we apply again Lemma 4.2.3 with $f = Q^\varepsilon$, (4.2.2) and (4.2.5), so that we also get:

$$\mathcal{E}_{M,\ell+\frac{1}{2},\delta}(Q^\varepsilon)(t) \leq \|Q^\varepsilon(0)\|^2_{\ell+\frac{1}{2},\delta} + C\Big(\varepsilon + (D^\varepsilon_{\ell-\frac{1}{2}})^2\Big)^{\frac{1}{2}} \|Q^\varepsilon(\tau)\|_{2,t,\ell+1,\delta}.$$

Therefore we have in the same way

$$\mathcal{E}_{2M_2,\ell+\frac{1}{2},\delta}(Q^\varepsilon)(t) \leq \|Q^\varepsilon(0)\|^2_{\ell+\frac{1}{2},\delta} + C\Big(\varepsilon + (D^\varepsilon_{\ell-\frac{1}{2}})^2\Big).$$

Hence, there holds

$$\mathcal{E}_{2M_2,\ell-\frac{1}{2},\delta}(P^\varepsilon)(t) + \mathcal{E}_{2M_2,\ell+\frac{1}{2},\delta}(Q^\varepsilon)(t) \leq (\tilde{D}^\varepsilon_{\ell-\frac{1}{2}})^2 + C(\varepsilon + (D^\varepsilon_{\ell-\frac{1}{2}})^2).$$

(4.2.1) yields

$$(\tilde{D}^\varepsilon_{\ell-\frac{1}{2}})^2 + (D^\varepsilon_{\ell-\frac{1}{2}})^2 = (\tilde{D}^\varepsilon_{\ell+\frac{1}{2}})^2,$$

so that

$$\mathcal{E}_{2M_2,\ell-\frac{1}{2},\delta}(P^\varepsilon)(t) + \mathcal{E}_{2M_2,\ell+\frac{1}{2},\delta}(Q^\varepsilon)(t) \leq C\Big(\varepsilon + (\tilde{D}^\varepsilon_{\ell+\frac{1}{2}})^2\Big).$$

The conclusion of the first statement then comes from the previous computation and the second part of Theorem 4.1.4 along with (4.2.1).

Case $\frac{d+1}{2} < \ell$. For this case, applying again Lemma 4.2.3 with $g = P^\varepsilon$ but with $m = \ell - 1 > \frac{d-1}{2}$, (4.2.10) and (4.2.12), the estimates give for some $C > 0$:

$$\mathcal{E}_{M,\ell-1,\delta}(P^\varepsilon)(t) \leq \|P^\varepsilon(0)\|^2_{\ell-1,\delta} + C(\varepsilon^2 + (D^\varepsilon_{\ell-1})^2)^{\frac{1}{2}} \|P^\varepsilon\|_{2,\ell-\frac{1}{2},\delta},$$

so that we obtain

$$\mathcal{E}_{2M_2,\ell-1,\delta}(P^\varepsilon)(t) \leq (\tilde{D}^\varepsilon_{\ell-1})^2 + C(\varepsilon^2 + (D^\varepsilon_{\ell-1})^2).$$

Then, for Q^ε, we apply Lemma 4.2.3, (4.2.2) and (4.2.5) with $f = Q^\varepsilon$, which gives:

$$\mathcal{E}_{M,\ell,\delta}(Q^\varepsilon)(t) \leq \|Q^\varepsilon(0)\|^2_{\ell,\delta} + C(\varepsilon^2 + (D^\varepsilon_{\ell-1})^2)^{\frac{1}{2}} \|Q^\varepsilon\|_{2,\ell+\frac{1}{2},\delta},$$

and the conclusion follows in the same way as in the previous case.

4.5. Semiclassical limit of the wave function

In this section, we address the semiclassical limit of the wave function $u^\varepsilon = e^{\frac{\psi^\varepsilon}{2} + \frac{\phi^\varepsilon}{\varepsilon}}$ with Theorem 4.1.14. We also prove Lemma 4.1.13. But first, we need to address the Cauchy problem of (4.1.19).

4.5.1. Cauchy problem of (4.1.19)

Theorem 4.5.1. *Let $\frac{d-1}{2} < m \leq l - 1$. For any $(\nabla\psi_{\text{in},1}, \nabla\phi_{\text{in},1}) \in \mathcal{H}^m_{\delta_{\text{in}}} \times \mathcal{H}^{m+1}_{\delta_{\text{in}}}$, there exists a unique $(\zeta_1, v_1) \in (L^\infty_T \mathcal{H}^m_\delta \times L^\infty_T \mathcal{H}^{m+1}_\delta) \cap (L^2_T \mathcal{H}^{m+\frac{1}{2}}_\delta \times L^2_T \mathcal{H}^{m+\frac{3}{2}}_\delta) \cap (\mathcal{C}_T \mathcal{H}^{m-\frac{1}{2}}_\delta \times \mathcal{C}_T \mathcal{H}^{m+\frac{1}{2}}_\delta)$ solution to* (4.1.19).

Proof. The proof is rather similar as the Cauchy theory of (4.1.15) developed in Section 4.2, and we present here the main steps. First, for the existence, we define a scheme:

— $\zeta_{1,0}(t) := \nabla\psi_{\text{in},1}$ and $v_{1,0}(t) := \nabla\phi_{\text{in},1}$ for all $t \geq 0$,

— For any $k \in \mathbb{N}$, $(\zeta_{1,k+1}, v_{1,k+1})$ is defined by

$$\begin{cases} \partial_t v_{1,k+1} + \nabla(v^0 \cdot v_{1,k+1}) + \lambda \operatorname{Re} \zeta_{1,k} = 0, \\ \partial_t \zeta_{1,k+1} + \nabla(v^0 \cdot \zeta_{1,k+1}) + \nabla(v_{1,k+1} \cdot \zeta^0) + \nabla \operatorname{div} v_{1,k+1} = \dfrac{i}{2}\Big(\nabla \operatorname{div} \zeta^0 + 2\nabla(\zeta^0 \cdot \zeta^0)\Big), \end{cases}$$

and the initial data $\zeta_{1,k+1}(0) = \nabla\psi_{\text{in},1}$ and $v_{1,k+1}(0) = \nabla\phi_{\text{in},1}$.

The previous system and Lemma 4.2.3 give the following estimates: for all $t \in [0, T]$ and $k \in \mathbb{N}$,

$$\mathcal{E}_{2M_2,m,\delta}(\zeta_{1,k+1})(t) + \mathcal{E}_{2M_2,m,\delta}(v_{1,k+1})(t) \leq |\lambda| |||\zeta_{1,k}|||_{2,T,m+\frac{1}{2},\delta} + \frac{1}{2}\sqrt{\omega_{\text{in}}} + K^\ell \omega_{\text{in}}.$$

From this estimate, one can prove by induction that, for all $k \in \mathbb{N}$ and $t \in [0, T]$,

$$\mathcal{E}_{2M_2,m,\delta}(\zeta_{1,k})(t) + \mathcal{E}_{2M_2,m,\delta}(v_{1,k})(t) \leq \sqrt{\omega_{\text{in}}} + 2K^\ell \omega_{\text{in}}.$$

Moreover, there also holds

$$\mathcal{E}_{2M_2,m,\delta}(\zeta_{1,k+1} - \zeta_{1,k})(t) + \mathcal{E}_{2M_2,m,\delta}(v_{1,k+1} - v_{1,k})(t) \leq |\lambda| |||\zeta_{1,k} - \zeta_{1,k-1}|||_{2,T,m+\frac{1}{2},\delta},$$

which proves the convergence of the scheme in $(\mathcal{C}_T \mathcal{H}^m_\delta \times \mathcal{C}_T \mathcal{H}^{m+1}_\delta) \cap (L^2_T \mathcal{H}^{m+\frac{1}{2}}_\delta \times L^2_T \mathcal{H}^{m+\frac{3}{2}}_\delta)$ like in Section 4.2.4. The uniqueness can also be proved with a similar computation. □

4.5.2. Proof of Theorem 4.1.14

We recall that $Z^\varepsilon = \zeta^\varepsilon - z^0$ and $V^\varepsilon = v^\varepsilon - v^0$. Define $\mathcal{Z}^\varepsilon := Z^\varepsilon - \varepsilon\zeta_1$ and $\mathcal{V}^\varepsilon := V^\varepsilon - \varepsilon v_1$. Then we have

$$\partial_t \mathcal{V}^\varepsilon + \nabla(v^0 \cdot \mathcal{V}^\varepsilon) + \lambda \operatorname{Re} \mathcal{Z}^\varepsilon = \frac{1}{2}\nabla(V^\varepsilon \cdot V^\varepsilon),$$
$$\partial_t \mathcal{Z}^\varepsilon + \nabla\Big(\mathcal{V}^\varepsilon \cdot \zeta^0\Big) + \nabla\Big(v^0 \cdot \mathcal{Z}^\varepsilon\Big) + \nabla \operatorname{div} \mathcal{V}^\varepsilon = \nabla\Big(V^\varepsilon \cdot Z^\varepsilon\Big) + i\frac{\varepsilon}{2}\Big(\nabla \operatorname{div} Z^\varepsilon + 2\nabla(Z^\varepsilon \cdot \zeta^0) + 2\nabla(Z^\varepsilon \cdot \zeta^\varepsilon)\Big).$$

This system is very similar to (4.3.1) and the estimates are actually the same up to two differences. First, the source terms (at the right-hand side of each equation) are $O(\varepsilon^2)$ but only in $L^2_T \mathcal{H}^{\ell-\frac{3}{2}}_\delta$ for the first equation and in $L^2_T \mathcal{H}^{\ell-\frac{5}{2}}_\delta$ for the second one. Indeed, there holds

$$|||\nabla(V^\varepsilon \cdot V^\varepsilon)|||_{2,T,\ell-\frac{3}{2},\delta} \leq K^\ell \sqrt{T} |||V^\varepsilon|||^2_{\infty,T,\ell,\delta} \leq C\varepsilon^2,$$

$$\|\|\nabla(V^\varepsilon \cdot Z^\varepsilon)\|\|_{2,T,\ell-\frac{5}{2},\delta} \leq K^{\ell-1}\sqrt{T}\|\|V^\varepsilon\|\|_{\infty,T,\ell-1,\delta}\|\|Z^\varepsilon\|\|_{\infty,T,\ell-1,\delta} \leq C\varepsilon^2,$$
$$\varepsilon\|\|\nabla \operatorname{div} Z^\varepsilon\|\|_{2,T,\ell-\frac{5}{2},\delta} \leq \varepsilon\|\|Z^\varepsilon\|\|_{2,T,\ell-\frac{1}{2},\delta} \leq C\varepsilon^2,$$
$$\varepsilon\Big\|\Big\|\nabla(Z^\varepsilon \cdot \zeta^0)\Big\|\Big\|_{2,T,\ell-\frac{5}{2},\delta} \leq \varepsilon K^{\ell-1}\sqrt{T}\|\|Z^\varepsilon\|\|_{\infty,T,\ell-1,\delta}\Big\|\Big\|\zeta^0\Big\|\Big\|_{\infty,T,\ell,\delta} \leq C\varepsilon^2,$$
$$\varepsilon\|\|\nabla(Z^\varepsilon \cdot \zeta^\varepsilon)\|\|_{2,T,\ell-\frac{5}{2},\delta} \leq \varepsilon K^{\ell-1}\sqrt{T}\|\|Z^\varepsilon\|\|_{\infty,T,\ell-1,\delta}\|\|\zeta^\varepsilon\|\|_{\infty,T,\ell,\delta} \leq C\varepsilon^2.$$

Then, we apply Lemma 4.2.3 with $m = \ell - 2$, and in particular (4.2.7). For this, we need $\ell - 2 > \frac{d-1}{2}$, which is assumed by Assumption 4. Therefore, we get for some $C > 0$ and for all $t \in [0, T]$

$$\mathcal{E}_{2M_2,\ell-2,\delta}(\mathcal{Z}^\varepsilon) + \mathcal{E}_{2M_2,\ell-1,\delta}(\mathcal{V}^\varepsilon)(t) \leq (r^\varepsilon_{\ell-2})^2 + C\,\varepsilon^4.$$

As for the case of ψ^ε and ϕ^ε, we also have $P^\varepsilon = \psi^\varepsilon - \psi^0$ and $Q^\varepsilon = \phi^\varepsilon - \phi^0$, so that by defining $\mathcal{P}^\varepsilon := P^\varepsilon - \varepsilon\psi_1$ and $\mathcal{Q}^\varepsilon = Q^\varepsilon - \varepsilon\phi_1$, we obtain

$$\partial_t \mathcal{P}^\varepsilon + \mathcal{V}^\varepsilon \cdot \zeta^0 + v^0 \cdot \mathcal{Z}^\varepsilon + \operatorname{div}\mathcal{V}^\varepsilon = V^\varepsilon \cdot Z^\varepsilon + i\frac{\varepsilon}{2}\Big(\operatorname{div} Z^\varepsilon + 2\,Z^\varepsilon \cdot \zeta^\varepsilon + 2\,Z^\varepsilon \cdot \zeta^0\Big),$$
$$\partial_t \mathcal{Q}^\varepsilon + \mathcal{V}^\varepsilon \cdot v^0 + \lambda \operatorname{Re} P^\varepsilon = \frac{1}{2}V^\varepsilon \cdot V^\varepsilon.$$

Like in Section 4.4.2 but with $m = \ell - 2$, we get with the first equation

$$\mathcal{E}_{M,\ell-2,\delta}(\mathcal{P}^\varepsilon)(t) \leq \|\mathcal{P}^\varepsilon(0)\|^2_{\ell-2,\delta} + C(\varepsilon^4 + (r^\varepsilon_{\ell-2})^2)^{\frac{1}{2}}\|P^\varepsilon\|_{2,\ell-\frac{3}{2},\delta},$$

and with the second one

$$\mathcal{E}_{M,\ell-1,\delta}(\mathcal{Q}^\varepsilon)(t) \leq \|\mathcal{Q}^\varepsilon(0)\|^2_{\ell-1,\delta} + C(\varepsilon^4 + (r^\varepsilon_{\ell-2})^2)^{\frac{1}{2}}\,\|Q^\varepsilon\|_{2,\ell-\frac{1}{2},\delta},$$

and the conclusion easily follows, using the fact that

$$\|\mathcal{P}^\varepsilon(0)\|^2_{\ell-2,\delta} + \|\mathcal{Q}^\varepsilon(0)\|^2_{\ell-1,\delta} + (r^\varepsilon_{\ell-2})^2 \leq 2(\tilde{r}^\varepsilon_{\ell-1})^2,$$

with (4.2.1).

4.5.3. Proof of Lemma 4.1.13

We now address Lemma 4.1.13. First, assume $\phi_1(t) \equiv 0$ on $[0, T]$, which yields not only $\phi_{\text{in},1} \equiv 0$ but also $\nabla\phi_1(t) \equiv 0$. Then, the first equation of (4.1.18) gives

$$\operatorname{Re}\psi_1 = 0.$$

This is in particular true for $t = 0$. Since $\psi_1(0) = \psi_{\text{in},1}$ is real-valued by assumption, we obtain $\psi_{\text{in},1} \equiv 0$.

On the other hand, since ϕ_1 is real-valued, taking the real part of the second equation of (4.1.18) leads to a system in $\operatorname{Re}\psi^1$ and ϕ^1:

$$\begin{cases} \partial_t\phi_1 + v^0 \cdot \nabla\phi_1 + \lambda \operatorname{Re}\psi_1 = 0, & \phi_1(0) = \phi_{\text{in},1}, \\ \partial_t \operatorname{Re}\psi_1 + v^0 \cdot \nabla \operatorname{Re}\psi_1 + \nabla\phi_1 \cdot \operatorname{Re}\zeta^0 + \Delta\phi_1 = 0, & \operatorname{Re}\psi_1(0) = \psi_{\text{in},1}. \end{cases}$$

This system is linear in $(\operatorname{Re}\psi_1, \phi_1)$, without any source term. Therefore, if $(\psi_{\text{in},1}, \phi_{\text{in},1}) \equiv (0, 0)$, we get $(\operatorname{Re}\psi_1, \phi_1) \equiv (0, 0)$, which gives the conclusion.

4.6. Assumptions on the initial data

In this section, we discuss about Assumption 1 for the initial data. In particular, the analytic behavior is asked only for the gradient of the initial data $(\nabla\psi^\varepsilon_{\text{in}}, \nabla\phi^\varepsilon_{\text{in}})$. One can show that this statement is different from asking the analyticity for the initial data $(\psi^\varepsilon_{\text{in}}, \phi^\varepsilon_{\text{in}})$ or even $(\psi^\varepsilon_{\text{in}}, \nabla\phi^\varepsilon_{\text{in}})$ directly. Indeed, when we consider the Fourier transform of these functions, we know that they are linked (for instance for $\psi^\varepsilon_{\text{in}}$) through the relation

$$\mathcal{F}(\nabla\psi^\varepsilon_{\text{in}}) = -i\xi\mathcal{F}(\psi^\varepsilon_{\text{in}}).$$

In particular, when we consider analyticity, we multiply these Fourier transforms by some $e^{-\delta_{\text{in}}\langle\xi\rangle}$ and ask them to be square integrable. Thus, if $\mathcal{F}(\nabla\psi^\varepsilon_{\text{in}})e^{-\delta_{\text{in}}\langle\xi\rangle}$ is L^2, then the previous relation gives that $\mathcal{F}(\psi^\varepsilon_{\text{in}})e^{-\delta_{\text{in}}\langle\xi\rangle}$ is square integrable for $|\xi| \to \infty$. However, we could still have a problem at $\xi = 0$, for instance if $|\mathcal{F}(\psi^\varepsilon_{\text{in}})(\xi)| \sim |\xi|^{-\frac{d}{2}}$ which is not square integrable but which gives $|\mathcal{F}(\nabla\psi^\varepsilon_{\text{in}})(\xi)|^2 \sim |\xi|^{-d+1}$ which is integrable.

This problem at $\xi = 0$ is actually linked to the behavior of $\psi_{\text{in}}^{\varepsilon}(x)$ for $|x| \to \infty$ since we formally have

$$\begin{aligned}\mathcal{F}(\nabla\psi_{\text{in}}^{\varepsilon})(0) &= \int \nabla\psi_{\text{in}}^{\varepsilon}(x)\,\mathrm{d}x \\ &= \lim_{R\to\infty} \int_{|x|\le R} \nabla\psi_{\text{in}}^{\varepsilon}(x)\,\mathrm{d}x \\ &= \lim_{R\to\infty} \int_{|x|=R} \psi_{\text{in}}^{\varepsilon}(y)\,\vec{n}\,\mathrm{d}\sigma(y).\end{aligned}$$

In particular, in dimension $d = 1$, we show that we can have any possible limit at infinity, and even different limits at $\pm\infty$.

Lemma 4.6.1. *For any pair $(a_-, a_+) \in (\mathbb{R} \cup \{\pm\infty\})^2$, there exists $f \in \mathcal{C}^1(\mathbb{R})$ such that $f' \in \mathcal{H}_\delta^0$ for any $\delta > 0$ and* $\lim_{\pm\infty} f = a_\pm$.

Proof. We will prove this result in three steps. First, we will prove it for $(a_-, a_+) \in \mathbb{R}^2$ by constructing a first function whose limit at $-\infty$ (resp. $+\infty$) is 0 (resp. 1) and satisfying the previous regularity conditions. Then, we will prove it when exactly one of them is finite and the other infinity. Finally, we will prove it for two infinity limits.

First step. We construct here a first function which will be used to prove the case $(a_-, a_+) \in \mathbb{R}^2$. Define

$$h_1(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

It is known that $h_1 \in L^1 \cap L^2$ and

$$\hat{h}_1(\xi) = e^{-\frac{\xi^2}{2}}.$$

Therefore, $h_1 \in \mathcal{H}_\delta^0$ for any $\delta > 0$. Then, we define:

$$g_1(x) := \int_{-\infty}^{x} h_1(y)\,\mathrm{d}y.$$

It is well defined since $h_1 \in L^1$. Moreover, it is obviously $\mathcal{C}^1$ with $g_1' = h_1 \in \mathcal{H}_\delta^0$ for any $\delta > 0$, and $\lim_{-\infty} g_1 = 0$. Furthermore, it is also known that

$$\lim_{+\infty} g_1 = \int_{-\infty}^{+\infty} h_1(y)\,\mathrm{d}y = 1.$$

Then, for $(a_-, a_+) \in \mathbb{R}^2$, $f_1 := a_- + (a_+ - a_-)\,g_1$ satisfies the needed assumptions.

Second step. We assume here $a_+ = +\infty$ and $a_- \in \mathbb{R}$. Define

$$h_2(x) := \begin{cases} (1+x)^{-1} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We know that $h_2 \in L^2$, therefore $\hat{h}_2 \in L^2$ too. However, $h_2 \notin L^1$. More precisely, h_2 is not integrable at $+\infty$. Then, we define g_2 by convolution:

$$g_2 := h_1 * h_2.$$

This is well defined pointwise since both h_1 and h_2 are in L^2, and it is also in L^2 since $h_1 \in L^1$. In particular, we have

$$\hat{g}_2(\xi) = \hat{h}_2(\xi)\,e^{-\frac{\xi^2}{2}} \in L^2(e^{-2\delta\langle\xi\rangle}\,\mathrm{d}\xi),$$

for every $\delta > 0$, therefore $g_2 \in \mathcal{H}_\delta^0$. In particular, $g_2 \in \mathcal{C}^\infty$. Moreover, $g_2 \ge 0$ from the fact that both h_1 and h_2 are non negative. Furthermore, g_2 has the same integrability property as h_2: it is integrable at $-\infty$ and is not at $+\infty$. Indeed, there holds

$$\begin{aligned}\int_{-\infty}^{0} g_2(x)\,\mathrm{d}x &= \int_{-\infty}^{0}\int_{-\infty}^{+\infty} h_2(x-y)\,h_1(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{-\infty}^{0}\int_{-\infty}^{x} h_2(x-y)\,h_1(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{-\infty}^{0}\int_{y}^{0} h_2(x-y)\,h_1(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{-\infty}^{0} h_1(y)\int_{y}^{0} h_2(x-y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

$$= \int_{-\infty}^{0} h_1(y) \int_{0}^{-y} h_2(x)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_{-\infty}^{0} h_1(y) \ln(1-y)\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

However, it is still not integrable:

$$\int_{-\infty}^{+\infty} g_2(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} h_1(x)\,\mathrm{d}x \int_{-\infty}^{+\infty} h_2(y)\,\mathrm{d}y = +\infty.$$

Hence, we can define the following function:

$$f_2(x) \coloneqq \int_{-\infty}^{x} g_2(y)\,\mathrm{d}y.$$

By definition, $f_2' = g_2 \in \mathcal{H}_\delta^0$ for all $\delta > 0$, $\lim_{-\infty} f = 0$ and $\lim_{+\infty} f = +\infty$. Hence, $a_- + f_2$ satisfies the needed properties. We can recover the other cases by adding a $-$ and/or considering $f_2(-x)$.

Third step. If both a_+ and a_- are $\pm\infty$, one can consider $f_3(x) = \pm f_2(x) \pm f_2(-x)$. □

Remark 4.6.2. In particular, there also holds $f' \in \mathcal{H}_\delta^\ell$ for any $\delta, \ell > 0$.

Remark 4.6.3. Even though this result is in dimension 1 for simplicity, the previous constructions can be extended to higher dimensions.

Chapitre 5

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