Non-Linear Vlasov Equation with Logarithmic Non-Linearity

M2 Thesis Defense

Guillaume Ferriere

Supervisors: Rémi Carles and Matthieu Hillairet

Institut Montpelliérain Alexandre Grothendieck (IMAG)
University of Montpellier

Orsay, 2018

Outline

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General Vlasov Equation

Vlasov Equation

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f + \mathbf{F}_0 \cdot \nabla_\xi f = 0.$$

with force term $\mathbf{F}_0 = \mathbf{F}_0(t, x, \xi)$

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- $\mathbf{F}_0$ (or V) may depend on f itself.

Logarithmic Vlasov Equation

Non-Linear Vlasov Equation with Logarithmic Non-Linearity

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f - \lambda \nabla_x (\ln \rho) \cdot \nabla_\xi f = 0, \qquad \text{(logVla)}$$

with $\lambda > 0$ and

$$\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,d\xi).$$

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Remark

- Non-linear
- Highly singular
- Formalization of the equation very difficult

Outline

Link with Euler and Schrödinger

Definition (Mono-Kinetic Measure)

A mono-kinetic measure is a measure of the form

$$f(t, dx, d\xi) = \rho(t, x)dx \otimes \delta_{\xi = v(t,x)},$$

with space distribution function ρ and speed v.

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A mono-kinetic measure is solution to (logVla) *iff* (ρ, v) *is solution of the Isothermal Euler System*

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho v) = 0, \\ \partial_t(\rho v) + \nabla_x \cdot (\rho v \otimes v) + \lambda \nabla_x \rho = 0. \end{cases} \tag{IES}$$

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Logarithmic Schrödinger Equation ($\epsilon > 0$)

$$i\epsilon\, \partial_t u_\epsilon + \frac{\epsilon^2}{2} \Delta u_\epsilon = \lambda\, u_\epsilon \ln |u_\epsilon|^2. \tag{logNLS$_\epsilon$}$$

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Wigner Transform

Definition (Wigner Transform [8, 7, 1, 6, 5])

For $u_\epsilon \in L^2(\mathbb{R}^d)$, the Wigner Transform W_ϵ is defined by

$$W_\epsilon(x,\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi\cdot z} u_\epsilon\left(x + \frac{\epsilon z}{2}\right) \overline{u_\epsilon\left(x - \frac{\epsilon z}{2}\right)} \, dz. \qquad \text{(WT)}$$

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- Reaches good results in order to perform the Semi-Classical Limit: if u_ϵ satisfies $i\epsilon\partial_t u_\epsilon + \frac{\epsilon^2}{2}\Delta u_\epsilon = V_0 u_\epsilon$ with V_0 satisfying suitable properties, then W verifies $\partial_t W + \xi \cdot \nabla_x W - \nabla_x V_0 \cdot \nabla_\xi W = 0$. Such a result also holds in some non-linear cases.

Outline

Solutions for the Logarithmic Schrödinger Equation

Theorem ([3, Theorem 1.5.])

Let $\lambda > 0$, $u_0 \in \mathcal{F}(H^1) \cap H^1(\mathbb{R}^d)$. Then there exists a unique, global solution $u \in L^\infty_{loc}(\mathbb{R}, \mathcal{F}(H^1) \cap H^1(\mathbb{R}^d))$ of

$$\begin{cases} i\,\partial_t u + \dfrac{\Delta u}{2} = \lambda u \ln |u|^2, \\ u(0,x) = u_0(x). \end{cases}$$

Moreover, $u \in \mathcal{C}(\mathbb{R}, L^2 \cap H^1_w(\mathbb{R}^d))$.

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Remark

This result can easily be generalized to the general case $\epsilon > 0$.

Universal Dynamics

Theorem ([3, Theorem 1.7.])

For $u_0 \neq 0$, *rescale the solution provided by the previous theorem to* $v = v(t,y)$ *by setting*

$$u(t,x) = \frac{1}{\tau(t)^{\frac{d}{2}}} \frac{||u_0||_{L^2}}{||\gamma||_{L^2}} v\left(t, \frac{x}{\tau(t)}\right) e^{i \frac{\dot{\tau}(t)}{\tau(t)} \frac{|x|^2}{2}},$$

where $\ddot{\tau} = \frac{2\lambda}{\tau}, \tau(0) = 1, \dot{\tau}(0) = 0$, *and* $\gamma(x) = e^{-\frac{|x|^2}{2}}$. *Then*

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} |v(t,y)|^2 \, dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \gamma^2(y) \, dy,$$

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Remark

$\tau(t) \underset{t\to\infty}{\sim} 2t\sqrt{\lambda \ln t}$: difference compared to the classical dispersion.

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Wigner Measure

Three questions:

- Does the Wigner Transform of u_ϵ solution to (logNLS$_\epsilon$) converge ?
- Is this limit a solution to the Logarithmic Vlasov Equation ?
- Does it have the same universal dynamics as previously said ?

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- Far from the vacuum: positive answer to the first two questions.

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The article of R. Carles and A. Nouri ([4]) goes along those intuitions in two cases:

- Far from the vacuum: positive answer to the first two questions.
- A class of explicit solutions for $d = 1$: the Gaussian case.
    - Explicit solutions to (logNLS$_\epsilon$).
    - Limit of the Wigner Transform : explicit mono-kinetic solution to (logVla).
    - Similar dispersion
    - Up to a rescaling, strong convergence of $\rho(t,x) = \int_{\mathbb{R}} W(t,x,d\xi)$ to $\gamma^2 = e^{-|x|^2}$ in $L^1(\mathbb{R})$.

Logarithmic Vlasov Equation

Two questions:

- Do any "solutions" to (logVla) formally have the same universal dynamics ?
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For the first question, we have two interesting cases:

- The previous Gaussian case. In the context of the Vlasov Equation, we call it the "Gaussian-monokinetic" case.
- A generalization of this case: the mono-kinetic case. This has been done by R. Carles, K. Carrapatoso and M. Hillairet in [2].

Outline

1. Introduction
2. Motivations
3. Universal Dynamics for the Wigner Measure
   - Main Theorem
   - Sketch of the proof
4. Universal Dynamics for the Logarithmic Vlasov Equation

Assumptions and notations

Assumptions

$$\lambda > 0, \qquad \rho_0 \geq 0, \qquad \sqrt{\rho_0} \in \mathcal{F}(H^1) \cap H^1(\mathbb{R}^d) \setminus \{0\}\,,$$
$$\phi_0 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d), \qquad \sqrt{\rho_0}\,\nabla\phi_0 \in L^2(\mathbb{R}^d), \tag{H1}$$

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- $u_{\epsilon,0} = \sqrt{\rho_0}\, e^{i\,\frac{\phi_0}{\epsilon}} \in \mathcal{F}(H^1) \cap H^1(\mathbb{R}^d) \setminus \{0\}.$

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- $$u_\epsilon(t,x) = \frac{1}{\tau(t)^{\frac{d}{2}}} \frac{||\sqrt{\rho_0}||_{L^2}}{||\gamma||_{L^2}}\, v_\epsilon\left(t, \frac{x}{\tau(t)}\right)\, e^{i\,\frac{\dot{\tau}(t)}{\tau(t)}\frac{|x|^2}{2\,\epsilon}},$$

  where we recall $\gamma(x) = e^{-\frac{|x|^2}{2}}$.

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  where we recall $\gamma(x) = e^{-\frac{|x|^2}{2}}$.
- W_ϵ (resp. $\tilde{W}_\epsilon$) the Wigner Transform of u_ϵ (resp. v_ϵ).

First properties

Proposition

Under the assumptions (H1) *and notations* (N1), *there exists* $\tilde{W} \in L^\infty((0,\infty), \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d))$ *such that, up to a subsequence,*

$$\tilde{W}_\epsilon \underset{\epsilon \to 0}{\rightharpoonup} \tilde{W} \qquad \text{in } L^1_{loc}((0,\infty), \mathcal{S}'_{w-*}(\mathbb{R}^d \times \mathbb{R}^d))$$

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Theorem (Integrability and regularity properties)

Under the assumptions (H1) *and notations* (N1)*, there holds for* $\tilde{W}$*:*

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \tilde{W}(t, dy, d\eta) = ||\gamma^2||_{L^1} \qquad \text{for a.e. } t \geq 0,$$

$$\tilde{\rho}(t,y) := \int_{\mathbb{R}^d} \tilde{W}(t, y, d\eta) \in \mathcal{C}(\mathbb{R}^+, W^{-1,1} \cap L^1_w(\mathbb{R}^d)),$$

$$\int_{\mathbb{R}^d} \tilde{\rho}(t,y) \left(|y|^2 + |\ln \tilde{\rho}(t,y)|\right) dy \leq C.$$

Main Theorem

Theorem (Universal Dynamics for a Wigner Measure of (logNLS$_\epsilon$))

Under the assumptions (H1) *and notations* (N1), *there holds for the limit* $\tilde{W}$ *of* $\tilde{W}_\epsilon$ *and* $\tilde{\rho} = \int_{\mathbb{R}^d} \tilde{W}(t, y, d\eta)$, *with* $\gamma(x) = e^{-\frac{|x|^2}{2}}$ *:*

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \end{pmatrix} \tilde{\rho}(t, y)\, dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \end{pmatrix} \gamma^2(y)\, dy,$$

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Remarks

- We still have the same dispersion rate in $(t\sqrt{\ln t})^{\frac{d}{2}}$.
- The lack of bounds of a higher moment in our proof does not allow us to reach the convergence of the quadratic momentum.

Outline

Integrability and regularity

$$\mathcal{E}^{\epsilon}_{\mathrm{kin}}(t) = \frac{\epsilon^2}{2\,\tau(t)^2} \| \nabla v_\epsilon \|^2_{L^2}, \qquad \mathcal{E}^{\epsilon}_{\mathrm{ent}}(t) = \int_{\mathbb{R}^d} |v_\epsilon(t,y)|^2 \ln \left| \frac{v_\epsilon(t,y)}{\gamma(y)} \right|^2,$$

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$$\mathcal{E}^{\epsilon}(t) = \mathcal{E}^{\epsilon}_{\text{kin}}(t) + \lambda\,\mathcal{E}^{\epsilon}_{\text{ent}}(t).$$

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Lemma

Under the assumptions (H1) and notations (N1), there holds

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End of the proof

$$\rho_\epsilon = |v_\epsilon|^2, \qquad J_\epsilon = \mathrm{Im}(\epsilon \, \overline{v_\epsilon} \, \nabla v_\epsilon).$$

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- We compute:

$$\partial_t \rho_\epsilon + \frac{1}{\tau^2(t)} \nabla \cdot J_\epsilon = 0,$$

$$\partial_t J_\epsilon + \lambda \nabla \rho_\epsilon + 2\lambda\, y\, \rho_\epsilon = \frac{\epsilon^2}{4\,\tau^2(t)} \Delta \nabla \rho - \frac{\epsilon^2}{\tau^2(t)} \nabla \cdot (\mathrm{Re}(\nabla v_\epsilon \otimes \overline{\nabla v_\epsilon})).$$

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- Passing the two previous equations to the limit $\epsilon \to 0$:

$$\partial_t \rho + \frac{1}{\tau^2(t)} \nabla \cdot J = 0,$$

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$$\rho_\epsilon = |v_\epsilon|^2, \qquad J_\epsilon = \mathrm{Im}(\epsilon\, \overline{v_\epsilon}\, \nabla v_\epsilon).$$

- We compute:

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- Change of time variable: $s = \frac{1}{2} \ln \tau(t)$.
- Unique weak limit $\tilde{\rho}_\infty = \gamma^2(y)$ of $\tilde{\rho}(s + .,.)$ when $s \to \infty$. □

Outline

Gaussian-Gaussian Solutions

Proposition (Gaussian-Gaussian solutions)

Solutions to (logVla) *of the form*

$$f(t,x,\xi) = \frac{1}{\pi\, c_1(t)\, c_2(t)} \; \exp\left[-\frac{|x-b_1(t)|^2}{c_1(t)^2} - \frac{|\xi-b_2(t,x)|^2}{c_2(t)^2}\right]$$

can be computed explicitly. Moreover, the functions c_i and b_i ($i=1,2$) are uniquely defined once the initial data (which can be reduced to 5 parameters) have been provided.

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Remark

- $c_1(t) \underset{t\to\infty}{\sim} 2\,t\sqrt{\lambda \ln t} \underset{t\to\infty}{\sim} \tau(t)$.
- Strong convergence to γ^2 in L^1 after rescaling.
- Other "generalization" of the Gaussian-monokinetic case.

Outline

Assumptions

- Mass conservation :

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} f(t,dx,d\xi)\right)=0, \tag{H2}$$

- Energy conservation :

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\xi|^2 f(t,dx,d\xi)+\lambda\int_{\mathbb{R}^d}\rho(t,x)\ln\rho(t,x)dx\right)=0, \tag{H3}$$

- Equations on ρ and J :

$$\partial_t\rho(t,x)+\nabla_x\cdot\left(\int_{\mathbb{R}^d}\xi\, f(t,x,d\xi)\right)=0, \tag{H4}$$

$$\partial_t\int_{\mathbb{R}^d}\xi\, f(t,x,d\xi)+\nabla_x\cdot\int_{\mathbb{R}^d}\xi\otimes\xi\, f(t,x,d\xi)+\lambda\,\nabla_x\rho(t,x)=0, \tag{H5}$$

Main result

Theorem (Universal Dynamics for the Logarithmic Vlasov Equation)

Assume that $f = f(t,x,\xi) \in L^\infty_{loc}((0,\infty); \mathcal{M}\Sigma_{\log} \cap \mathcal{M}_2 \setminus \{0\})$ *satisfies* (H2)-(H5). *Rescale:*

$$f(t,x,\xi) = \frac{M}{\|\gamma^2\|_{L_1}} \tilde{f}\left(t, \frac{x}{\tau(t)}, \tau(t)\xi - \dot{\tau}(t)x\right),$$

where M *is the total mass. Then*

$$\tilde{\rho} \in L^\infty((0,\infty), L^1_2 \cap L\log L(\mathbb{R}^d)) \cap \mathcal{C}(\mathbb{R}^+, L^1_w(\mathbb{R}^d)),$$

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \tilde{\rho}(t,y)\,dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \gamma^2(y)\,dy,$$

$$\tilde{\rho}(t,.) \underset{t\to\infty}{\rightharpoonup} \gamma^2 \qquad \text{weakly in } L^1(\mathbb{R}^d).$$

Summary

- The Wigner Transform of the solutions of (logNLS$_\epsilon$) converges and the dynamics of the limit is universal, similar to the universal dynamics found for (logNLS$_\epsilon$), with a weak convergence to γ^2 in L^1.
- Universal dynamics are proven in the same way for the Logarithmic Vlasov Equation by assuming some formal properties, completed by a new class of explicit Gaussian solutions whose convergence is strong.

Summary

- The Wigner Transform of the solutions of (logNLS$_\epsilon$) converges and the dynamics of the limit is universal, similar to the universal dynamics found for (logNLS$_\epsilon$), with a weak convergence to γ^2 in L^1.
- Universal dynamics are proven in the same way for the Logarithmic Vlasov Equation by assuming some formal properties, completed by a new class of explicit Gaussian solutions whose convergence is strong.

Perspectives:

- Convergence to γ^2: uniform in ϵ ? In Wasserstein distance ?
- Analytic initial data: mono-kinetic Wigner Measure for small time ?
- The link between Wigner Measure and Vlasov equation still needs to be proven rigorously.

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